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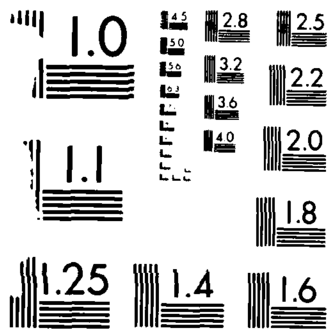

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COUPLING PROBLEMS WITH HIGH BIREFRINGENCE FIBERS

prepared by

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Coupling Problems with High Birefringence Fibers

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Abstract

A program to investigate the coupling problems of high birefringence fibers was initiated in 1983 for Naval Research Laboratory under Contract N00014-83-K-2029. The technical monitor was Dr. W. K. Burns of Naval Research Laboratory. The work performed under this contract is summarized in this report.

A fiber is birefringent if it has a non-circular core and/or cladding, or is subjected to anisotropic stress. Currently, the loss of fibers with embedded anisotropic stress is much lower than that with non-circular core or cladding and therefore these fibers are of interest. Central to any coupling problem is the distribution of fields in the fibers. To date, very little is known relative to the fields of polarization modes in biaxial fibers. This is the main thrust of our study. We have used a perturbation method to study the fields and dispersion of polarization modes in biaxial fibers. In our approach, the inclusion or exclusion of a term is determined by the differential equation, the boundary conditions and the symmetry of the problem, and does not rely on the prior knowledge of the fields in the fibers. The dependence of the fiber birefringence on the index differential and index anisotropy has been studied. Now that the fields in high birefringent fiber are known, the excitation of high birefringent fibers can be analyzed. Methods for studying these excitation problems are indicated and some of the results are presented.

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1. Introduction

Since low-loss fibers were realized some two decades ago, interest on fibers has been evolved from multimode fibers to single mode fibers. In reality, all conventional single mode fibers support two orthogonal polarization modes. As these polarization modes are nearly degenerate, fields in these fibers are easily converted from one polarization mode to the another if there is slightly disturbance or perturbation. For sophisticated systems like coherent communication systems, sensory systems based on interferometric principles, or systems utilizing polarization-dependent components, these changes in the state of polarization would lead to signal fading and noise. The polarization mode conversion, being the result of mode coupling, can be greatly reduced if the polarization mode degeneracy is removed [1,2]. With polarization mode degeneracy removed, fibers become highly birefringent since their propagation velocities are quite different. If the birefringence is sufficiently large, and the fiber core is sufficiently small, only one polarization mode can be guided by the fiber. Then we have single-mode single-polarization fibers.

That the polarization modes in the conventional single-mode fibers are nearly degenerate can be traced to the fact that these fibers have : (i) nominally circular cross sections, (ii) isotropic index and (iii) azimuthally independent index profile. When one or more of these conditions is removed, the polarization mode becomes non-degenerate. The simplest and most obvious examples are fibers with noncircular cross sections (and isotropic and ϕ-independent index profiles). Fibers with either an elliptical core, cladding or both are of these category and have been studied extensively by many authors [3-6].

Fibers can also be made nondegenerate by making the core, cladding or both anisotropic. If either region is biaxial, a fiber becomes nondegenerate even if it has a perfectly circular cross section. Through mechanical stress, isotropic materials can be

made anisotropic. The stress can be built into the fibers by the fabrication processes or applied after fabrication. Of particular interest is index anisotropy induced by lateral stress. Although fibers with uniaxial index of refraction have been the subject of several studies [7-9], fibers with biaxial material have not been examined until recently [10-21]. Despite of these studies, fibers under lateral stress are not very well understood. This is the subject of this work. Recently, a series of papers on high birefringence fibers have been published by Snyder and his associates [16-21]. However, there is a basic difference between their work and the work reported here. In their work, they have assumed from the outset that LP^x_{10} mode is coupled with LP^y_{21} mode, and LP^y_{10} mode with LP^x_{21} mode. Here, no assumption has been made relative to the coupling between any polarization modes. All modes are included in the consideration. The inclusion or exclusion of a particular mode is determined by the differential equation, the boundary conditions and the symmetry of the problem, and does not rely on our prior knowledge of the field distribution in the fiber.

In the following, we begin by reviewing the effects of the stress on the refractive index. Under lateral stress,an isotropic medium becomes biaxial [22,23]. Therefore waves guided by biaxial media are also reviewed. Complication associated with biaxial media are noted and ways to solve the problem are described. This is done in Sections 2 and 3. We then proceed to study the fields and dispersion of fibers with biaxial core and cladding (Section 4). The numerical results are presented in Section 5. Also discussed are the similarities and differences between elliptical fibers and biaxial fibers. The coupling of biaxial fibers is discussed in the last section. The differences between low- and high- birefringent fibers, so far as coupling is concerned, is also noted there.

2. Waves Guided by Uniaxial and Biaxial Media

When a solid is stressed, its refractive index changes and this is known as the photoelastic effect. Consider an isotropic material with an index n under stress-free condition. When stressed, its indices along principal stress directions become:

$$n_x \simeq n + [C_1 T_{xx} + C_2(T_{yy} + T_{zz})]$$

$$n_y \simeq n + [C_1 T_{yy} + C_2(T_{zz} + T_{xx})]$$

$$n_z \simeq n + [C_1 T_{zz} + C_2(T_{xx} + T_{yy})]$$

where T_{xx}, T_{yy}, and T_{zz} are the stress components in Cartesian coordinates and C_1 and C_2 are the stress-optic constants. These stress-optic constants are negative and are of the order of 10^{-12} m^2/N in MKS system. More importantly, $|C_2|$ is a few times larger than $|C_1|$ [22,23].

Since the fiber core and cladding regions are made of different material, the thermal expansion coefficients for two regions are likely different. After fibers are drawn from the preforms and allowed to cool down, lateral stress is frozen in the fibers. This is the basis of built-in lateral stress in many polarization-holding fibers [24,25 and 26]. If in addition, the fiber cross section is not circular and we have $T_{xx} \neq T_{yy}$. In the absence of any build-in axial stress ($T_{zz} = 0$), the above equations become:

$$n_x \simeq n + [C_1 T_{xx} + C_2 T_{yy}] \simeq n_z - (C_2 - C_1)T_{xx}$$

$$n_y \simeq n + [C_1 T_{yy} + C_2 T_{xx}] \simeq n_z - (C_2 - C_1)T_{yy}$$

$$n_z \simeq n + [C_2(T_{xx} + T_{yy})]$$

Clearly, the core and cladding regions are biaxial with $n_z \neq n_x \neq n_y$.

To study the waves guided by fibers with biaxial core and/or cladding, it would be necessary to examine the waves guided by biaxial media [27,28]. For this purpose, we begin with the time-harmonic Maxwell's equations for biaxial media :

$$\nabla \times \vec{E}(x,y,z) = j\omega\mu_o \vec{H}(x,y,z) \tag{1}$$

$$\nabla \times \vec{H}(x,y,z) = -j\omega \epsilon_o \overleftrightarrow{\epsilon} \cdot \vec{E}(x,y,z) \tag{2}$$

$$\nabla \cdot \vec{H}(x,y,z) = 0 \tag{3}$$

$$\nabla \cdot \epsilon_o \overleftrightarrow{\epsilon} \cdot \vec{E}(x,y,z) = 0. \tag{4}$$

A time-harmonic variation of the form $e^{-j\omega t}$ is understood. In a matrix form therelative dielectric tensor $\overleftrightarrow{\epsilon}$ may be written as

$$\overleftrightarrow{\epsilon} = \begin{bmatrix} n_x^2 & 0 & 0 \\ 0 & n_y^2 & 0 \\ 0 & 0 & n_z^2 \end{bmatrix} \tag{5}$$

When the geometry of the guiding structure and the relative dielectric tensor are independent of z and waves propagating along z direction are of interest, a term $e^{+j\beta z}$ can be factored out explicitly. Here β is the propagation constant of the guided mode. It is also convenient to separate the field components and the ∇ operator into transverse and longitudinal parts:

$$\vec{E}(x,y,z) = [\, \vec{e}_t(x,y) + \hat{a}_z\, e_z(x,y)\,]e^{j\beta z} \tag{6a}$$

$$\vec{H}(x,y,z) = [\, \vec{h}_t(x,y) + \hat{a}_z\, h_z(x,y)\,]e^{j\beta z} \tag{6b}$$

$$\nabla = \nabla_t + j\beta\, \hat{a}_z = [\hat{a}_x \frac{\partial}{\partial x} + \hat{a}_y \frac{\partial}{\partial y}] + j\beta\, \hat{a}_z \tag{7}$$

where $\hat{a}_z$ is a unit vector in $+z$ direction and the subscript t signifies the components transverse to the z direction. In terms of these variables and ∇_t, (1) and (2) become

$$\nabla_t \times \vec{e}_t = j\omega\mu_o h_z \hat{a}_z \tag{8}$$

$$\hat{a}_z \times (j\beta\vec{e}_t - \nabla_t e_z) = j\omega\mu_o \vec{h}_t \tag{9}$$

$$\nabla_t \times \vec{h}_t = -\, j\omega\epsilon_o\, \epsilon_z e_z \hat{a}_z \tag{10}$$

$$\hat{\mathbf{a}}_z \times (j\beta\vec{\mathbf{h}}_t - \nabla_t \mathbf{h}_z) = -j\omega\,\epsilon_o \overleftrightarrow{\epsilon}_t \cdot \vec{\mathbf{e}}_t \tag{11}$$

where

$$\overleftrightarrow{\epsilon}_t = \begin{bmatrix} n_x^2 & 0 \\ 0 & n_y^2 \end{bmatrix} \tag{12}$$

By eliminating $\vec{\mathbf{h}}_t$ or $\vec{\mathbf{e}}_t$ from (9) and (11), we obtain

$$\vec{\mathbf{e}}_t = j\beta\mathbf{K}\cdot\nabla_t \mathbf{e}_z - j\omega\mu_o\mathbf{K}\cdot\hat{\mathbf{a}}_z \times \nabla_t \mathbf{h}_z \tag{13}$$

$$\vec{\mathbf{h}}_t = j\omega\,\epsilon_o\,\hat{\mathbf{a}}_z \times \mathbf{K}\cdot\overleftrightarrow{\epsilon}_t\cdot\nabla_t \mathbf{e}_z - j\beta\,\hat{\mathbf{a}}_z \times \mathbf{K}\cdot\hat{\mathbf{a}}_z \times \nabla_t\,\mathbf{h}_z \tag{14}$$

where

$$\mathbf{K} = \begin{bmatrix} K_x^{-2} & 0 \\ 0 & K_y^{-2} \end{bmatrix} \tag{15}$$

$$K_x^2 = k_o^2\,n_x^2 - \beta^2$$

$$K_y^2 = k_o^2\,n_y^2 - \beta^2$$

and $k_o^2 = \omega^2\,\mu_o\,\epsilon_o$. Differential equations for $\mathbf{e}_z$ and $\mathbf{h}_z$ can be obtained by substituting (13) - (14) into (3) and (4):

$$n_x^2 K_y^2 \frac{\partial^2 \mathbf{e}_z}{\partial x^2} + n_y^2 K_x^2 \frac{\partial^2 \mathbf{e}_z}{\partial y^2} + K_x^2 K_y^2\,n_z^2\,\mathbf{e}_z = \omega\mu_o\beta(n_x^2 - n_y^2)\,\frac{\partial^2 \mathbf{h}_z}{\partial x \partial y} \tag{16}$$

$$K_x^2 \frac{\partial^2 \mathbf{h}_z}{\partial x^2} + K_y^2 \frac{\partial^2 \mathbf{h}_z}{\partial y^2} + K_x^2 K_y^2\,\mathbf{h}_z = \omega\epsilon_o\beta(n_x^2 - n_y^2)\,\frac{\partial^2 \mathbf{e}_z}{\partial x \partial y} \tag{17}$$

Once $\mathbf{e}_z$ and $\mathbf{h}_z$ are known, (13) and (14) can be used to evaluate the transverse field components.

For isotropic as well as uniaxial media ($n_x = n_y$), clearly $K_x = K_y$ and $\mathbf{e}_z$ and $\mathbf{h}_z$ are uncoupled and (16) and (17) become the usual wave equations for $\mathbf{e}_z$ and $\mathbf{h}_z$. In this

fashion , the dispersion relation for uniaxial fibers can be and has been derived [7,8,9]. However, the differences between isotropic and uniaxial fibers are noteworthy. For isotropic fibers, the transverse electric fields of LP_{01} mode is mainly along one direction, x or y, and $|\vec{e}_t| / |\vec{h}_t|$ is approximately a constant. In uniaxial fibers, on the other hand, both $\mathbf{e}_x$ and $\mathbf{e}_y$ are present. Each components of $\vec{e}_t$ and $\vec{h}_t$ have to be expressed as the sum of two parts, one involves the ordinary index of refraction, $n_x = n_y = n_o$ and the other the extraordinary index of refraction $n_z = n_e$. As a result, $|\vec{e}_t| / |\vec{h}_t|$ is not a constant.

For biaxial media, $n_x^2 \neq n_y^2 \neq n_z^2$, $\mathbf{e}_z$ and $\mathbf{h}_z$ are coupled in general. Two-dimensional waveguide structures, integrated planar waveguides for example, are exceptions. For the infinitely large, two dimensional structures we can further assume $\frac{\partial}{\partial x} = 0$ or $\frac{\partial}{\partial y} = 0$, and (16) and (17) again can be reduced to the usual uncoupled wave equations for $\mathbf{e}_z$ and $\mathbf{h}_z$. This is precisely the approach adapted by Burns and Warner in their study of uniaxial planar waveguide structures [24]. No such simplification is possible for the fibers with biaxial media and herein lies the crux of the problem. In addition, K_x^2 and K_y^2 can be positive or negative depending on the values of β, $k_o n_x$ and $k_o n_y$. Thus (16) and (17) can change from differential equations of elliptical type to that of parabolic type as β changes. Thus for biaxial media, (16) and (17) are inherently difficult to solve, and no exact solution is known.

Alternatively, $\mathbf{e}_z$, $\mathbf{h}_z$ and $\vec{h}_t$ can be expressed in terms of $\vec{e}_t$. From (4) and (6a) we have

$$\mathbf{e}_z = \frac{j}{\beta \epsilon_z} \nabla_t \cdot (\vec{\epsilon}_t \cdot \vec{e}_t) \tag{18}$$

By performing vector product of ∇_t on both sides of (8), we solve for $\hat{\mathbf{a}}_z \times \nabla_t \mathbf{h}_z$. Similarly, (9) is used to solve for $\hat{\mathbf{a}}_z \times \vec{h}_t$. Upon substituting the expressions so obtained into (11) a differential equation for $\vec{e}_t$ is obtained:

$$\nabla_t^2 \vec{e}_t + (k_o^2 \overleftrightarrow{\epsilon}_t \cdot - \beta^2)\vec{e}_t = \nabla_t[(\nabla_t \cdot \vec{e}_t) - \frac{1}{\epsilon_z}\nabla_t \cdot (\overleftrightarrow{\epsilon}_t \cdot \vec{e}_t)] \tag{19a}$$

Similarly, an differential equation for $\mathbf{h}_t$ is also obtained:

$$\nabla_t^2 \vec{\mathbf{h}}_t + (k_o^2 \epsilon_z + \beta^2 \epsilon_z \hat{\mathbf{a}}_z \times \overleftrightarrow{\epsilon}_t^{-1} \cdot \hat{\mathbf{a}}_z \times)\vec{\mathbf{h}}_t$$

$$= \nabla_t(\nabla_t \times \vec{\mathbf{h}}_t) - j\,\beta\,\epsilon_z\,\hat{\mathbf{a}}_z \times \overleftrightarrow{\epsilon}_t^{-1} \cdot \hat{\mathbf{a}}_z \times \nabla_t \mathbf{h}_z \tag{19b}$$

In component form (19a) can be written as:

$$\frac{\partial^2 \mathbf{e}_x}{\partial x^2} + \frac{n_z^2}{n_x^2}\frac{\partial^2 \mathbf{e}_x}{\partial y^2} + k_o^2 \frac{n_z^2}{n_x^2}(n_x^2 - N^2)\,\mathbf{e}_x = \frac{n_z^2 - n_y^2}{n_x^2}\frac{\partial^2 \mathbf{e}_y}{\partial x \partial y} \tag{20a}$$

$$\frac{n_z^2}{n_y^2}\frac{\partial^2 \mathbf{e}_y}{\partial x^2} + \frac{\partial^2 \mathbf{e}_y}{\partial y^2} + k_o^2 \frac{n_z^2}{n_y^2}(n_y^2 - N^2)\,\mathbf{e}_y = \frac{n_z^2 - n_x^2}{n_y^2}\frac{\partial^2 \mathbf{e}_x}{\partial x \partial y} \tag{20b}$$

where $N = \beta/k_o$ is the effective index of refraction of the guided mode.

Although $\mathbf{e}_x$ and $\mathbf{e}_y$ are still coupled in (20a) and (20b), the differential equations are simplified. In particular, for the fibers of interest, $|n_x^2 - n_y^2|/n_x^2 << 1$, the terms in the right hand side of (20a) and (20b) are small and can be treated as perturbation terms. Then approximate solutions can be obtained from (20a) and (20b). This will be done in Section 4. However, some features of the solution can be deduced from (20a) and (20b) without actually solving the differential equations. Clearly if $\mathbf{e}_x$ is an even [or odd] function of x or y, then $\mathbf{e}_y$ must be odd [or even] function of x or y. When expressed in terms of cylindrical coordinates $\mathbf{r},\phi$, $\mathbf{e}_x$ and $\mathbf{e}_y$ can be written as Fourier sine and cosine series. Thus for the core region ($r \leq a$),

$$\mathbf{e}_x(r,\phi) = e_{x0}^R(r) + \sum_{i=1}^{\infty}[e_{xi}^{Rc}(r)\cos i\phi + e_{xi}^{Rs}(r)\sin i\phi] \tag{21a}$$

$$\mathbf{e}_y(r,\phi) = e_{y0}^R(r) + \sum_{i=1}^{\infty}[e_{yi}^{Rc}(r)\cos i\phi + e_{yi}^{Rs}(r)\sin i\phi] \tag{21b}$$

and for the cladding region ($r > a$):

$$\mathbf{e}_x(r,\phi) = e_{x0}^{D}(r) + \sum_{i=1}^{\infty}[e_{xi}^{Dc}(r)\cos i\phi + e_{xi}^{Ds}(r)\sin i\phi] \tag{22a}$$

$$\mathbf{e}_y(r,\phi) = e_{y0}^{D}(r) + \sum_{i=1}^{\infty}[e_{yi}^{Dc}(r)\cos i\phi + e_{yi}^{Ds}(r)\sin i\phi] \tag{22b}$$

The subscripts R and D are used to signify that these terms are associated with the core and cladding regions respectively. From (20a) and (20b) it is clear that the sine and cosine series of $\mathbf{e}_x$ are coupled respectively with the cosine and sine series of $\mathbf{e}_y$. In addition, the even harmonics are coupled only with even harmonics, and odd harmonics with odd harmonics only.

3. Fields and Boundary Conditions for Weakly-guiding Fibers

An isotropic fiber, with core and cladding indices n_r and n_d respectively, is viewed as "weakly guiding" if $n_r - n_d << n_r$. Under the condition of weakly guiding, $|\nabla_t \mathbf{e}_z|$ and $|\nabla_t \mathbf{h}_z|$, are negligibly small in comparison with $|\beta \vec{\mathbf{e}}_t|$ and $|\beta \vec{\mathbf{h}}_t|$, and therefore can be neglected. Then from (9) and (11) we have:

$$\vec{\mathbf{h}}_t \simeq \frac{\omega\epsilon_0}{\beta} n_r^2\, \hat{\mathbf{a}}_z \times \vec{\mathbf{e}}_t \tag{23}$$

which is accurate to the order of $|(n_r - n_d)/n_r|$. Thus for weakly-guiding fibers, $\vec{\mathbf{e}}_t$ and $\vec{\mathbf{h}}_t$ are related in the same manner as that of uniform plane waves in isotropic media [29]. This is a basic feature of fields of weakly guiding isotropic fiber.

Now consider weakly guiding anisotropic fibers. In the core region, the indices along three principal directions are n_{rx}, n_{ry} and n_{rz}. Correspondingly, the indices in the cladding region are n_{dx}, n_{dy} and n_{dz}. The core radius is a. By weakly guiding, it is meant that $|(n_{rx} - n_{dx})/n_{rx}|$ etc. are small and are of the order of $O(\epsilon)$. Under these conditions, following the same argument presented by Gloge for isotropic fibers [29], we

have

$$\vec{\mathbf{h}}_t \simeq \frac{\omega\epsilon_o}{\beta}\, \hat{\mathbf{a}}_z \times \overleftrightarrow{\epsilon}_t \cdot \vec{\mathbf{e}}_t \tag{24}$$

With $\vec{\mathbf{e}}_t$ known, exact expressions for $\mathbf{h}_z$ and $\mathbf{e}_z$ and approximate expression for $\vec{\mathbf{h}}_t$ can be obtained from (9), (18) and (24).

It has been noted in the last section that the cosine series [and sine series] of $\mathbf{e}_x$ and sine series [and cosine series] of $\mathbf{e}_y$ are coupled through the differential equations (20a) and (20b). We like to show that the same conclusion can also be reached by considering the boundary conditions. To enforce the boundary conditions, it is expedient to express $\vec{\mathbf{e}}_t$ and $\vec{\mathbf{h}}_t$ in terms as $\mathbf{e}_r$, $\mathbf{e}_\phi$, $\mathbf{h}_r$ and $\mathbf{h}_\phi$. In terms of the coefficients introduced in (21a) and (21b) the field components in the core region are:

$$\begin{aligned}
2\mathbf{e}_\phi = {} & (e_{y1}^{Rc} - e_{x1}^{Rs}) \\
& + (e_{x2}^{Rc} - 2e_{x0}^{R} + e_{y2}^{Rs}) \sin\phi + (e_{y2}^{Rc} + 2e_{y0}^{R} - e_{x2}^{Rs}) \cos\phi \\
& + \sum_{j=2}^{\infty} [-e_{x\,j-1}^{Rc} + e_{x\,j+1}^{Rc} + e_{y\,j-1}^{Rs} + e_{y\,j+1}^{Rs}] \sin j\phi \\
& + \sum_{j=2}^{\infty} [e_{x\,j-1}^{Rs} - e_{x\,j+1}^{Rs} + e_{y\,j-1}^{Rc} + e_{y\,j+1}^{Rc}] \cos j\phi
\end{aligned} \tag{25}$$

$$\begin{aligned}
-j2\beta\mathbf{e}_z = {} & \frac{n_{rx}^2}{n_{rz}^2}[e_{x1}^{Rc\prime} + \frac{1}{r}\, e_{x1}^{Rc}] + \frac{n_{ry}^2}{n_{rz}^2}[e_{y1}^{Rs\prime} + \frac{1}{r}\, e_{y1}^{Rs}] \\
& + [\frac{n_{rx}^2}{n_{rz}^2}(2e_{x0}^{R\prime} + e_{x2}^{Rc\prime} + \frac{2}{r}e_{x2}^{Rc}) + \frac{n_{ry}^2}{n_{rz}^2}(e_{y2}^{Rs\prime} + \frac{2}{r}e_{y2}^{Rs})]\cos\phi \\
& + [\frac{n_{rx}^2}{n_{rz}^2}(e_{x2}^{Rs\prime} + \frac{2}{r}e_{x2}^{Rs}) + \frac{n_{ry}^2}{n_{rz}^2}(2e_{y0}\mathrm{spR}' - e_{y2}^{Rc\prime} - \frac{2}{r}e_{y2}^{Rc})]\sin\phi
\end{aligned}$$

$$+\sum_{j=2}^{\infty}\Big[\ \frac{n_{rx}^2}{n_{rz}^2}(e_{xj-1}^{Rs}{}' + e_{xj+1}^{Rs}{}' - \frac{j-1}{r}e_{xj-1}^{Rs} + \frac{j+1}{r}e_{xj+1}^{Rs})$$

$$+\ \frac{n_{ry}^2}{n_{rz}^2}(e_{yj-1}^{Rc}{}' - e_{yj+1}^{Rc}{}' - \frac{j-1}{r}e_{yj-1}^{Rc} - \frac{j+1}{r}e_{yj+1}^{Rc})\Big]\sin j\phi$$

$$+\sum_{j=2}^{\infty}\Big[\ \frac{n_{rx}^2}{n_{rz}^2}(e_{xj-1}^{Rc}{}' + e_{xj+1}^{Rc}{}' + \frac{j+1}{r}e_{xj+1}^{Rc} - \frac{j-1}{r}e_{xj-1}^{Rc})$$

$$+\ \frac{n_{ry}^2}{n_{rz}^2}(e_{yj+1}^{Rs}{}' - e_{yj-1}^{Rs}{}' + \frac{j-1}{r}e_{yj-1}^{Rs} + \frac{j+1}{r}e_{yj+1}^{Rs})\Big]\cos j\phi \tag{26}$$

$$j2\omega\mu_o r\mathbf{h}_z = r(e_{y1}^{Rc\prime} - e_{x1}^{Rs\prime}) + (e_{y1}^{Rc} - e_{x1}^{Rs})$$

$$+\left\{ r(e_{x2}^{Rc\prime} - 2e_{x0}^{R\prime} + e_{y2}^{Rs\prime}) + 2(e_{x2}^{Rc} + e_{y2}^{Rs}) \right\}\sin\phi$$

$$+\left\{ r(e_{y2}^{Rc\prime} + 2e_{y0}^{R\prime} - e_{x2}^{Rs\prime}) + 2(e_{y2}^{Rc} - e_{x2}^{Rs}) \right\}\cos\phi$$

$$+\sum_{j=2}^{\infty}\Big\{ r[-e_{x\,j-1}^{Rc}{}' + e_{x\,j+1}^{Rc}{}' + e_{y\,j-1}^{Rs}{}' + e_{y\,j+1}^{Rs}{}']$$

$$+\ [(j-1)e_{x\,j-1}^{Rc} + (j+1)e_{x\,j+1}^{Rc} - (j-1)e_{y\,j-1}^{Rs} + (j+1)e_{y\,j+1}^{Rs}]\Big\}\sin j\phi$$

$$+\sum_{j=2}^{\infty}\Big\{ r[e_{x\,j-1}^{Rs}{}' - e_{x\,j+1}^{Rs}{}' + e_{y\,j-1}^{Rc}{}' + e_{y\,j+1}^{Rc}{}']$$

$$+\ [-(j-1)e_{x\,j-1}^{Rs} - (j+1)e_{x\,j+1}^{Rs} - (j-1)e_{y\,j-1}^{Rc} + (j+1)e_{y\,j+1}^{Rc}]\Big\}\cos j\phi$$

$$\tag{27}$$

$$\frac{2\beta}{\omega\epsilon_o}\mathbf{h}_\phi \simeq [n_{rx}^2 e_{x1}^{Rc} + n_{ry}^2 e_{y1}^{Rs}]$$

$$+[n_{rx}^2 e_{x2}^{Rc} + 2n_{rx}^2 e_{x0}^{R} + n_{ry}^2 e_{y2}^{Rs}] \cos\phi + [n_{rx}^2 e_{x2}^{Rs} + 2n_{ry}^2 e_{y0}^{R} - n_{ry}^2 e_{y2}^{Rc}] \sin\phi$$

$$+\sum_{j=2}^{\infty}[n_{rx}^2 e_{x\,j-1}^{Rc} + n_{rx}^2 e_{x\,j+1}^{Rc} - n_{ry}^2 e_{y\,j-1}^{Rs} + n_{ry}^2 e_{y\,j+1}^{Rs}]\cos j\phi$$

$$+\sum_{j=2}^{\infty}[n_{rx}^2 e_{x\,j-1}^{Rs} + n_{rx}^2 e_{x\,j+1}^{Rs} + n_{ry}^2 e_{y\,j-1}^{Rc} - n_{ry}^2 e_{y\,j+1}^{Rc}]\sin j\phi \tag{28}$$

In these expressions a prime indicates $\partial/\partial r$. Corresponding expressions can be written for the field components in the cladding region, with the superscript R replaced by D and n_{rx} etc. by n_{dx} etc.

At the core-cladding interface ($r = a$), $\mathbf{e}_\phi$, $\mathbf{e}_z$, $\mathbf{d}_r$, $\mathbf{h}_\phi$, $\mathbf{h}_z$ and $\mathbf{b}_r$ should be continous. It is known that when the continuity for the tangential components of $\vec{\mathbf{E}}$ and $\vec{\mathbf{H}}$ are satisfied, the continuity for the normal components for $\vec{\mathbf{D}}$ and $\vec{\mathbf{B}}$ are automatically satisfied [30] Since sin $j\phi$ and cos $j\phi$ are orthogonal functions in the range $(0, 2\pi)$. the continuation of the tangential components of $\vec{\mathbf{E}}$ and $\vec{\mathbf{H}}$ amounts to the continuation of the individual Fourier components. More specifically,

$$e_{y1}^{Rc} - e_{x1}^{Rs} = e_{y1}^{Dc} - e_{x1}^{Ds} \tag{29a}$$

$$\frac{n_{rx}^2}{n_{rz}^2}[e_{x1}^{Rc\prime} + \frac{1}{a} e_{x1}^{Rc}] + \frac{n_{ry}^2}{n_{rz}^2}[e_{y1}^{Rs\prime} + \frac{1}{a} e_{y1}^{Rs}]$$

$$= \frac{n_{dx}^2}{n_{dz}^2}[e_{x1}^{Dc\prime} + \frac{1}{a} e_{x1}^{Dc}] + \frac{n_{dy}^2}{n_{dz}^2}[e_{y1}^{Ds\prime} + \frac{1}{a} e_{y1}^{Ds}] \tag{29b}$$

$$n_{rx}^2 e_{x1}^{Rc} + n_{ry}^2 e_{y1}^{Rs} = n_{dx}^2 e_{x1}^{Dc} + n_{dy}^2 e_{y1}^{Ds} \tag{29c}$$

$$a(e_{y1}^{Rc\prime} - e_{x1}^{Rs\prime}) + (e_{y1}^{Rc} - e_{x1}^{Rs}) = a(e_{y1}^{Dc\prime} - e_{x1}^{Ds\prime}) + (e_{y1}^{Dc} - e_{x1}^{Ds}) \tag{29d}$$

$$e_{x2}^{Rc} - 2e_{x0}^{R} + e_{y2}^{Rs} = e_{x2}^{Dc} - 2e_{x0}^{D} + e_{y2}^{Ds} \tag{30a}$$

$$e_{y2}^{Rc} + 2e_{y0}^{R} - e_{x2}^{Rs} = e_{y2}^{Dc} + 2e_{y0}^{D} - e_{x2}^{Ds} \tag{30b}$$

$$\frac{n_{rx}^2}{n_{rz}^2}(2e_{x0}^{R\prime} + e_{x2}^{Rc\prime} + \frac{2}{a}e_{x2}^{Rc}) + \frac{n_{ry}^2}{n_{rz}^2}(e_{y2}^{Rs\prime} + \frac{2}{a}e_{y2}^{Rs})$$

$$= \frac{n_{dx}^2}{n_{dz}^2}(2e_{x0}^{D\prime} + e_{x2}^{Dc\prime} + \frac{2}{a}e_{x2}^{Dc}) + \frac{n_{dy}^2}{n_{dz}^2}(e_{y2}^{Ds\prime} + \frac{2}{a}e_{y2}^{Ds}) \tag{30c}$$

$$\frac{n_{rx}^2}{n_{rz}^2}(e_{x2}^{Rs\prime} + \frac{2}{a}e_{x2}^{Rs}) + \frac{n_{ry}^2}{n_{rz}^2}(2e_{y0}^{R\prime} - e_{y2}^{Rc\prime} - \frac{2}{a}e_{y2}^{Rc})$$

$$= \frac{n_{dx}^2}{n_{dz}^2}(e_{x2}^{Ds\prime} + \frac{2}{a}e_{x2}^{Ds}) + \frac{n_{dy}^2}{n_{dz}^2}(2e_{y0}^{D\prime} - e_{y2}^{Dc\prime} - \frac{2}{a}e_{y2}^{Dc}) \tag{30d}$$

$$n_{rx}^2e_{x2}^{Rc} + 2n_{rx}^2e_{x0}^{R} + n_{ry}^2e_{y2}^{Rs} = n_{dx}^2e_{x2}^{Dc} + 2n_{dx}^2e_{x0}^{D} + n_{dy}^2e_{y2}^{Ds} \tag{30e}$$

$$n_{rx}^2e_{x2}^{Rs} + 2n_{ry}^2e_{y0}^{R} - n_{ry}^2e_{y2}^{Rc} = n_{dx}^2e_{x2}^{Ds} + 2n_{dy}^2e_{y0}^{D} - n_{dy}^2e_{y2}^{Dc} \tag{30f}$$

$$a(e_{x2}^{Rc\prime} - 2e_{x0}^{R\prime} + e_{y2}^{Rs\prime}) + 2(e_{x2}^{Rc} + e_{y2}^{Rs})$$

$$= a(e_{x2}^{Dc\prime} - 2e_{x0}^{D\prime} + e_{y2}^{Ds\prime}) + 2(e_{x2}^{Dc} + e_{y2}^{Ds}) \tag{30g}$$

$$a(e_{y2}^{Rc\prime} + 2e_{y0}^{R\prime} - e_{x2}^{Rs\prime}) + 2(e_{y2}^{Rc} - e_{x2}^{Rs})$$

$$= a(e_{y2}^{Dc\prime} + 2e_{y0}^{D\prime} - e_{x2}^{Ds\prime}) + 2(e_{y2}^{Dc} - e_{x2}^{Ds}) \tag{30h}$$

And for $j \geq 2$,

$$-e_{x\,j-1}^{Rc} + e_{x\,j+1}^{Rc} + e_{y\,j-1}^{Rs} + e_{y\,j+1}^{Rs} = -e_{x\,j-1}^{Dc} + e_{x\,j+1}^{Dc} + e_{y\,j-1}^{Ds} + e_{y\,j+1}^{Ds} \quad \textbf{(31a)}$$

$$e_{x\,j-1}^{Rs} - e_{x\,j+1}^{Rs} + e_{y\,j-1}^{Rc} + e_{y\,j+1}^{Rc} = e_{x\,j-1}^{Ds} - e_{x\,j+1}^{Ds} + e_{y\,j-1}^{Dc} + e_{y\,j+1}^{Dc} \quad \textbf{(31b)}$$

$$\Big[\ \frac{n_{rx}^2}{n_{rz}^2}\Big(e_{xj-1}^{Rc}{}' + e_{xj+1}^{Rc}{}' + \frac{j+1}{a}e_{xj+1}^{Rc} - \frac{j-1}{a}e_{xj-1}^{Rc}\Big)$$

$$+ \frac{n_{ry}^2}{n_{rz}^2}\Big(e_{yj+1}^{Rs}{}' - e_{yj-1}^{Rs}{}' + \frac{j-1}{a}e_{yj-1}^{Rs} + \frac{j+1}{a}e_{yj+1}^{Rs}\Big)\Big]$$

$$= \Big[\ \frac{n_{dx}^2}{n_{dz}^2}\Big(e_{xj-1}^{Dc}{}' + e_{xj+1}^{Dc}{}' + \frac{j+1}{a}e_{xj+1}^{Dc} - \frac{j-1}{a}e_{xj-1}^{Dc}\Big)$$

$$+ \frac{n_{dy}^2}{n_{dz}^2}\Big(e_{yj+1}^{Ds}{}' - e_{yj-1}^{Ds}{}' + \frac{j-1}{a}e_{yj-1}^{Ds} + \frac{j+1}{a}e_{yj+1}^{Ds}\Big)\Big] \quad \textbf{(31c)}$$

$$\Big[\ \frac{n_{rx}^2}{n_{rz}^2}\Big(e_{xj-1}^{Rs}{}' + e_{xj+1}^{Rs}{}' - \frac{j-1}{a}e_{xj+1}^{Rs} + \frac{j+1}{a}e_{xj+1}^{Rs}\Big)$$

$$+ \frac{n_{ry}^2}{n_{rz}^2}\Big(e_{yj-1}^{Rc}{}' - e_{yj+1}^{Rc}{}' - \frac{j-1}{a}e_{yj-1}^{Rc} - \frac{j+1}{a}e_{yj+1}^{Rc}\Big)\Big]$$

$$= \Big[\ \frac{n_{dx}^2}{n_{dz}^2}\Big(e_{xj-1}^{Ds}{}' + e_{xj+1}^{Ds}{}' - \frac{j-1}{a}e_{xj+1}^{Ds} + \frac{j+1}{a}e_{xj+1}^{Ds}\Big)$$

$$+ \frac{n_{dy}^2}{n_{dz}^2}\Big(e_{yj-1}^{Dc}{}' - e_{yj+1}^{Dc}{}' - \frac{j-1}{a}e_{yj-1}^{Dc} - \frac{j+1}{a}e_{yj+1}^{Dc}\Big)\Big] \quad \textbf{(31d)}$$

$$n_{rx}^2 e_{x\,j-1}^{Rc} + n_{rx}^2 e_{x\,j+1}^{Rc} - n_{ry}^2 e_{y\,j-1}^{Rs} + n_{ry}^2 e_{y\,j+1}^{Rs}$$

$$= n_{dx}^2 e_{x\,j-1}^{Dc} + n_{dx}^2 e_{x\,j+1}^{Dc} - n_{dy}^2 e_{y\,j-1}^{Ds} + n_{dy}^2 e_{y\,j+1}^{Ds} \quad \textbf{(31e)}$$

$$n_{rx}^2 e_{x\,j-1}^{Rs} + n_{rx}^2 e_{x\,j+1}^{Rs} + n_{ry}^2 e_{y\,j-1}^{Rc} - n_{ry}^2 e_{y\,j+1}^{Rc}$$

$$= n_{dx}^2 e_{x\,j-1}^{Ds} + n_{dx}^2 e_{x\,j+1}^{Ds} + n_{dy}^2 e_{y\,j-1}^{Dc} - n_{dy}^2 e_{y\,j+1}^{Dc} \tag{31f}$$

$$a[-e_{x\,j-1}^{Rc}{}' + e_{x\,j+1}^{Rc}{}' + e_{y\,j-1}^{Rs}{}' + e_{y\,j+1}^{Rs}{}']$$

$$+ [(j-1)e_{x\,j-1}^{Rc} + (j+1)e_{x\,j+1}^{Rc} - (j-1)e_{y\,j-1}^{Rs} + (j+1)e_{y\,j+1}^{Rs}]$$

$$= a[-e_{x\,j-1}^{Dc}{}' + e_{x\,j+1}^{Dc}{}' + e_{y\,j-1}^{Ds}{}' + e_{y\,j+1}^{Ds}{}']$$

$$+ [(j-1)e_{x\,j-1}^{Dc} + (j+1)e_{x\,j+1}^{Dc} - (j-1)e_{y\,j-1}^{Ds} + (j+1)e_{y\,j+1}^{Ds}] \tag{31g}$$

$$a[e_{x\,j-1}^{Rs}{}' - e_{x\,j+1}^{Rs}{}' + e_{y\,j-1}^{Rc}{}' + e_{y\,j+1}^{Rc}{}']$$

$$+ [-(j-1)e_{x\,j-1}^{Rs} - (j+1)e_{x\,j+1}^{Rs} - (j-1)e_{y\,j-1}^{Rc} + (j+1)e_{y\,j+1}^{Rc}]$$

$$= a[e_{x\,j-1}^{Ds}{}' - e_{x\,j+1}^{Ds}{}' + e_{y\,j-1}^{Dc}{}' + e_{y\,j+1}^{Dc}{}']$$

$$+ [-(j-1)e_{x\,j-1}^{Ds} - (j+1)e_{x\,j+1}^{Ds} - (j-1)e_{y\,j-1}^{Dc} + (j+1)e_{y\,j+1}^{Dc}] \tag{31h}$$

Examination of these boundary condition equations reveals that the cosine series of $\mathbf{e}_x$ are coupled with the sine series of $\mathbf{e}_y$, and the sine series of $\mathbf{e}_x$ with the cosine series of $\mathbf{e}_y$. Furthermore, the even harmonics are coupled only with the even harmonics, and odd harmonics with odd harmonics. For example, in (30a)-(30d), e_{x0}^R and e_{x0}^D are coupled with e_{x2}^{Rc}, e_{y2}^{Rs} e_{x2}^{Dc} and e_{y2}^{Ds} and through these terms to the higher order even harmonics.

In the case of isotropic fibers, e_{x0}^R and e_{x0}^D [or e_{y0}^R and e_{y0}^D] are needed to determine the fields and dispersion of LP_{01} mode, No higher order terms is required. It would be interesting to see if the same is also true for biaxial fibers. Suppose that only one term from each series is kept, (i.e., e_{x0}^R for the core region and e_{x0}^D for the cladding region). Then, the boundary conditions (30a), (30c),(30e) and (30g) become

$$e^R_{x0} = e^D_{x0} \tag{30a1}$$

$$\frac{n^2_{rx}}{n^2_{rz}} e^{R\prime}_{x0} = \frac{n^2_{dx}}{n^2_{dz}} e^{D\prime}_{x0} \tag{30c1}$$

$$n^2_{rx}\, e^R_{x0} = n^2_{dx}\, e^D_{x0} \tag{30e1}$$

$$e'^{\,R}_{x0} = e'^{\,D}_{x0} \tag{30g1}$$

For isotropic fibers, $n_{rz} = n_{rx} = n_r$ and $n_{dz} = n_{dx} = n_d$, (30c1) is identical to (30g1). Recall the expression for $\mathbf{e}_\phi$ is exact while that for $\mathbf{h}_\phi$ is approximate and accurate to $O(\epsilon)$. When terms of the order of $O(\epsilon)$ or smaller are ignored, (30e1) also reduces to (30a1). Therefore in the case of weakly guided isotropic fibers, the boundary conditions are simply the continuation of e_{x0} and e'_{x0}

For biaxial fibers, $n_{rx} \neq n_{ry} \neq n_{rz}$ (30c1) and (30g1) are not identical. The presence of n^2_{rx}/n^2_{rz} etc. makes it impossible to satisfy all boundary conditions if e^R_{x0} and e^D_{x0} are the only terms kept. There have to be the higher harmonic terms.

Suppose only the leading harmonic term of cosine series of $\mathbf{e}_x$ (e^R_{x0} and e^D_{x0}) and the sine series of $\mathbf{e}_y$ (e^{Rs}_{y2} and e^{Ds}_{y2}) are kept. There are two unknown Fourier coefficients for the fields in the core region and two in the cladding region. In addition the propagation constant β is also an unknown. Four equations [(30a), (30c), (30e) and (30g)] are used to eliminate four unknowns. The result is a dispersion relation determining β or the effective index N. If desired, we can use the first two or first three harmonic terms of the sine and cosine series, then we have, respectively 8 or 12 unknown coefficients. Thus 8 or 12 boundary condition equations respectively are needed to set up the dispersion relation for β. By solving the dispersion relation, we can determine β as a function of various fiber parameters. In addition all but one unknown Fourier coefficients can be expressed in terms of one Fourier coefficient. The undetermined Fourier coefficient can be determined from the field strength of the mode or the power

carried by the fields.

4. Approximate Expressions of e_x and e_y and Dispersion Relation

To derive the approximate expressions for the transverse electric field components, we treat the right-hand terms of (20a) and (20b), as perturbation terms. Thus, by ignoring the terms on right-hand side of (20a) and (20b), we obtain the zeroth-order solutions, $e_x^{(0)}$ and $e_y^{(0)}$. With the zero-th order solutions substituted in the right-hand side terms of (20a) and (20b) and treated as known quantities, then (20a) and (20b) become inhomogeneous differential equations. The particular solutions for these differential equations are then combined with the homogeneous solutions which are exactly $e_x^{(0)}$ and $e_y^{(0)}$. Thus we have $e_x^{(1)}$ and $e_y^{(1)}$ which are accurate to $O(\epsilon)$.

Two lowest order polarization modes are mutually orthogonal to each other. In one polarization mode, its e_x is described by a cosine series and e_y by a sine series. For the other polarization mode, e_x and e_y are given by sine and cosine series respec vely. In this section, our attention is focused on one of the polarization modes, namely the mode with a cosine series for e_x and a sine series for e_y. Since no restriction is placed on the value of n_{rx} relative to n_{ry}, or n_{dx} relative to n_{dy}. The other polarization mode, can be obtained simply by interchanging n_{rx} and n_{dx} with n_{ry} and n_{dy}.

Fields in the core region:

To calculate the fields in the core region, we introduce two new variables, $x'' = (n_{ry}/n_{rz})x$, and $y' = (n_{rx}/n_{rz})y$. Thus (20a) and (20b) become:

$$\frac{\partial^2 e_x}{\partial x^2} + \frac{\partial^2 e_x}{\partial y'^2} + \frac{n_{rz}^2}{n_{rx}^2} k_o^2 (n_{rx}^2 - N^2)\, e_x = \frac{n_{rz}^2 - n_{ry}^2}{n_{rx} n_{rz}} \frac{\partial^2 e_y}{\partial x \partial y'} \tag{32a}$$

$$\frac{\partial^2 \mathbf{e}_y}{\partial x''^2} + \frac{\partial^2 \mathbf{e}_y}{\partial y^2} + \frac{n_{rz}^2}{n_{ry}^2} k_o^2 (n_{ry}^2 - N^2)\, \mathbf{e}_y = \frac{n_{rz}^2 - n_{rx}^2}{n_{ry} n_{rz}} \frac{\partial^2 \mathbf{e}_x}{\partial x'' \partial y} \tag{32b}$$

When the right-hand side terms in (32a) and (32b) are ignored, we have immediately the zero-order solutions:

$$\mathbf{e}_x^{(0)} = A_0\, J_0\left(\frac{n_{rz}}{n_{rx}} U_{rx} \frac{r'}{a}\right) + \sum_1^{\infty} A_m\, J_m\left(\frac{n_{rz}}{n_{rx}} U_{rx} \frac{r'}{a}\right) \cos m\phi' \tag{33}$$

$$\mathbf{e}_y^{(0)} = \sum_1^{\infty} B_m\, J_m\left(\frac{n_{rz}}{n_{ry}} U_{ry} \frac{r''}{a}\right) \sin m\phi'' \tag{34}$$

where

$$U_{rx} = k_o a \sqrt{n_{rx}^2 - N^2} \tag{35a}$$

$$U_{ry} = k_o a \sqrt{n_{ry}^2 - N^2} \tag{35b}$$

$$r' = [x^2 + y'^2]^{-1/2} = [x^2 + \frac{n_{rx}^2}{n_{rz}^2} y^2]^{-1/2}, \qquad \phi' = \tan^{-1}\left(\frac{n_{rx} y}{n_{rz} x}\right) \tag{35c}$$

$$r'' = [x''^2 + y^2] = [\frac{n_{ry}^2}{n_{rz}^2} x^2 + y^2]^{-1/2}, \qquad \phi'' = \tan^{-1}\left(\frac{n_{rz} y}{n_{ry} x}\right) \tag{35d}$$

As stated previously, only cosine and sine series are kept in (33) and (34) respectively. In defining the new variables y' or x'', the coordinates are scaled by constant factors but there is no translation in the origin. Thus the symmetry properties of $\mathbf{e}_x$ and $\mathbf{e}_y$ with respect to x'', y', ϕ' or ϕ'' remain the same as those with respect to x and y, and ϕ, as discussed in Section 3.

With $\mathbf{e}_y^{(0)}$ and $\mathbf{e}_x^{(0)}$ known and substituted into (32a) and (32b) respectively, (32a) and (32b) become a set of inhomogeneous differential equations. Their solutions are comprised of two parts: a homogeneous solution and a particular solution. The homogeneous solutions are of the same form as the zeroth-order solution. In view of the

functional form of $\mathbf{e}_x^{(0)}$ and $\mathbf{e}_y^{(0)}$, we can expect that the particular solutions can also be expressed as series of $J_m(\frac{n_{rz}}{n_{ry}} U_{ry} \frac{r}{a}) \cos m\,\phi$ or $J_m(\frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a}) \sin m\,\phi$. By substituting these series into the left-hand side of (32a) and (32b), performing the necessary differentiation, and comparing the resulting terms with those on the right-side, the particular solutions are found. In the lengthy manipulation, the recurrence relations for Bessel functions [31] are used repeatly and we also note in particular that $U_{rx} \neq U_{ry}$. Then $\mathbf{e}_x^{(1)}$ and $\mathbf{e}_y^{(1)}$, accurate to $O(\epsilon)$, are obtained by combining the the particular solutions with the zero-th order terms:

$$\mathbf{e}_x^{(1)} = A_0\, J_0(\frac{n_{rz}}{n_{rx}} U_{rx} \frac{r'}{a}) + \sum_1^{\infty} A_m\, J_m(\frac{n_{rz}}{n_{rx}} U_{rx} \frac{r'}{a}) \cos m\,\phi'$$

$$+ \frac{1}{4} \frac{n_{ry}(n_{rz}^2 - n_{ry}^2)}{n_{rz}(n_{rx}^2 - n_{ry}^2)} \frac{n_{ry}^2 - N^2}{N^2}$$

$$\sum_1^{\infty} B_m \left[J_{m-2}(\frac{n_{rz}}{n_{ry}} U_{ry} \frac{r}{a}) \cos (m-2)\phi - J_{m+2}(\frac{n_{rz}}{n_{ry}} U_{ry} \frac{r}{a}) \cos (m+2)\phi \right] \tag{36}$$

$$\mathbf{e}_y^{(1)} = \sum_1^{\infty} B_m\, J_m(\frac{n_{rz}}{n_{ry}} U_{ry} \frac{r''}{a}) \sin m\phi''$$

$$+ \frac{1}{4} \frac{n_{rx}(n_{rz}^2 - n_{rx}^2)}{n_{rz}(n_{ry}^2 - n_{rx}^2)} \frac{n_{rx}^2 - N^2}{N^2} \{ 2A_0\, J_2(\frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a}) \sin 2\phi$$

$$+ \sum_1^{\infty} A_m \left[J_{m+2}(\frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a}) \sin (m+2)\phi - J_{m-2}(\frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a}) \sin (m-2)\phi \right] \} \tag{37}$$

Note that $\left| n_{rx}(n_{rz}^2-n_{rx}^2)/[n_{rz}(n_{ry}^2-n_{rx}^2)] \right|$ and $\left| n_{ry}(n_{rz}^2-n_{ry}^2)/[n_{rz}(n_{rx}^2-n_{ry}^2)] \right|$ are of the order of O(1), while $\left|(n_{rx}^2-N^2)/N^2\right|$ and $\left|(n_{ry}^2-N^2)/N^2\right|$ are of the order of $O(\epsilon)$.

Fields in the cladding region:

Again, new variables may be introduced to simplify the differential equation. In particular r', ϕ', r'' and ϕ'' are defined in the same fashion as given in (35c) and (35d) with n_{rx} etc. substituted by n_{dx} etc. For isotropic fibers, N of a guided mode is always larger than n_d. For biaxial fibers, complication arises particularly for modes near cutoff since $n_{dx} \neq n_{dy}$. Near cutoff, it is possible to have $n_{dx} < N < n_{dy}$ or $n_{dx} > N > n_{dy}$ depending on the relative values of n_{dx} and n_{dy}. Thus three cases have to be treated separately. For convenience, we define:

$$W_{dx} = k_o a \sqrt{N^2 - n_{dx}^2} \tag{38a}$$

$$W_{dy} = k_o a \sqrt{N^2 - n_{dy}^2} \tag{38b}$$

$$U_{dx} = k_o a \sqrt{n_{dx}^2 - N^2} \tag{38c}$$

$$U_{dy} = k_o a \sqrt{n_{dy}^2 - N^2} \tag{38d}$$

For $N > n_{dx}$ and n_{dy}, W_{dx}^2 and W_{dy}^2 are positive. Following the same procedure described in previous section, we obtain:

$$e_x^{(1)} = C_0 K_0\left(\frac{n_{dz}}{n_{dx}} W_{dx} \frac{r'}{a}\right) + \sum_1^{\infty} C_m K_m\left(\frac{n_{dz}}{n_{dx}} W_{dx} \frac{r'}{a}\right) \cos m\phi'$$

$$- \frac{1}{4} \frac{n_{dy}(n_{dz}^2-n_{dy}^2)}{n_{dz}(n_{dx}^2-n_{dy}^2)} \frac{n_{dy}^2-N^2}{N^2}$$

$$\sum_1^\infty D_m \left[K_{m-2}\left(\frac{n_{dz}}{n_{dy}} W_{dy} \frac{r}{a}\right) \cos(m-2)\phi - K_{m+2}\left(\frac{n_{dz}}{n_{dy}} W_{dy} \frac{r}{a}\right) \cos(m+2)\phi \right]$$

(39)

$$e_y^{(1)} = \sum_1^\infty D_m K_m\left(\frac{n_{dz}}{n_{dy}} W_{dy} \frac{r''}{a}\right) \sin m\phi''$$

$$- \frac{1}{4} \frac{n_{dx}(n_{dz}^2-n_{dx}^2)}{n_{dz}(n_{dy}^2-n_{dx}^2)} \frac{n_{dx}^2-N^2}{N^2} \{ 2C_0 K_2\left(\frac{n_{dz}}{n_{dx}} W_{dx} \frac{r}{a}\right) \sin 2\phi$$

$$+ \sum_1^\infty C_m \left[K_{m+2}\left(\frac{n_{dz}}{n_{dx}} W_{dx} \frac{r}{a}\right) \sin(m+2)\phi - K_{m-2}\left(\frac{n_{dz}}{n_{dx}} W_{dx} \frac{r}{a}\right) \sin(m-2)\phi\right]\}$$

(40)

If $n_{dx} > n_{dy}$, there exists situations where $n_{dx} > N > n_{dy}$. With $W_{dx}^2 > 0$ and $U_{dy}^2 > 0$, the fields in the cladding are described by a mixture of the Hankel function and the modified Bessel function. The presence of the Hankel function signifies the existence of the leaky waves in the cladding region.

$$e_x^{(1)} = C_0 K_0\left(\frac{n_{dz}}{n_{dx}} W_{dx} \frac{r'}{a}\right) + \sum_1^\infty C_m K_m\left(\frac{n_{dz}}{n_{dx}} W_{dx} \frac{r'}{a}\right) \cos m\phi'$$

$$+ \frac{1}{4} \frac{n_{dy}(n_{dz}^2-n_{dy}^2)}{n_{dz}(n_{dx}^2-n_{dy}^2)} \frac{n_{dy}^2-N^2}{N^2}$$

$$\sum_1^\infty D_m \left[H_{m-2}^{(2)}\left(\frac{n_{dz}}{n_{dy}} U_{dy} \frac{r}{a}\right) \cos(m-2)\phi - H_{m+2}^{(2)}\left(\frac{n_{dz}}{n_{dy}} U_{dy} \frac{r}{a}\right) \cos(m+2)\phi \right]$$

(41)

$$e_y^{(1)} = \sum_1^{\infty} D_m \, H_m^{(2)}\!\left(\frac{n_{dz}}{n_{dy}} \, U_{dy} \, \frac{r''}{a}\right) \sin m\phi''$$

$$- \frac{1}{4} \, \frac{n_{dx}(n_{dz}^2 - n_{dx}^2)}{n_{dz}(n_{dy}^2 - n_{dx}^2)} \, \frac{n_{dx}^2 - N^2}{N^2} \Big\{ 2C_0 \, K_2\!\left(\frac{n_{dz}}{n_{dx}} \, W_{dx} \, \frac{r}{a}\right) \sin 2\phi$$

$$+ \sum_1^{\infty} C_m \left[K_{m+2}\!\left(\frac{n_{dz}}{n_{dx}} \, W_{dx} \, \frac{r}{a}\right) \sin (m+2)\phi - K_{m-2}\!\left(\frac{n_{dz}}{n_{dx}} \, W_{dx} \, \frac{r}{a}\right) \sin (m-2)\phi \right] \Big\}$$

$$(42)$$

On the other hand, if $n_{dx} < n_{dy}$, there may be situations where $U_{dx}^2 > 0$ and $W_{dy}^2 > 0$. Then the fields are given by:

$$e_x^{(1)} = C_0 \, H_0^{(2)}\!\left(\frac{n_{dz}}{n_{dx}} \, U_{dx} \, \frac{r'}{a}\right) + \sum_1^{\infty} C_m \, H_m^{(2)}\!\left(\frac{n_{dz}}{n_{dx}} \, U_{dx} \, \frac{r'}{a}\right) \cos m\phi'$$

$$- \frac{1}{4} \, \frac{n_{dy}(n_{dz}^2 - n_{dy}^2)}{n_{dz}(n_{dx}^2 - n_{dy}^2)} \, \frac{n_{dy}^2 - N^2}{N^2}$$

$$\sum_1^{\infty} D_m \left[K_{m-2}\!\left(\frac{n_{dz}}{n_{dy}} \, W_{dy} \, \frac{r}{a}\right) \cos (m-2)\phi - K_{m+2}\!\left(\frac{n_{dz}}{n_{dy}} \, W_{dy} \, \frac{r}{a}\right) \cos (m+2)\phi \right]$$

$$(43)$$

$$e_y^{(1)} = \sum_1^{\infty} D_m \, K_m\!\left(\frac{n_{dz}}{n_{dy}} \, W_{dy} \, \frac{r''}{a}\right) \sin m\phi''$$

$$+ \frac{1}{4} \, \frac{n_{dx}(n_{dz}^2 - n_{dx}^2)}{n_{dz}(n_{dy}^2 - n_{dx}^2)} \, \frac{n_{dx}^2 - N^2}{N^2} \Big\{ 2C_0 \, H_2^{(2)}\!\left(\frac{n_{dz}}{n_{dx}} \, U_{dx} \, \frac{r}{a}\right) \sin 2\phi$$

$$+ \sum_1^{\infty} C_m \left[H_{m+2}^{(2)}\!\left(\frac{n_{dz}}{n_{dx}} \, U_{dx} \, \frac{r}{a}\right) \sin (m+2)\phi - H_{m-2}^{(2)}\!\left(\frac{n_{dz}}{n_{dx}} \, U_{dx} \, \frac{r}{a}\right) \sin (m-2)\phi \right] \Big\}$$

(44)

Further approximations:

Before imposing the boundary conditions, it is necessary to revert r', r'', ϕ' and ϕ'' back to r and ϕ. For points in the core region,

$$r' = [x^2 + \frac{n_{rx}^2}{n_{rz}^2} y^2]^{-1/2} \simeq r - \frac{r}{4}(1 - \frac{n_{rx}^2}{n_{rz}^2})(1 - \cos 2\phi) \tag{45a}$$

$$r'' = [\frac{n_{ry}^2}{n_{rz}^2} x^2 + y^2]^{-1/2} \simeq r - \frac{r}{4}(1 - \frac{n_{ry}^2}{n_{rz}^2})(1 + \cos 2\phi) \tag{45b}$$

$$\phi' \simeq \phi - \frac{1}{2}(1 - \frac{n_{rx}}{n_{rz}}) \sin 2\phi \tag{45c}$$

and,

$$\phi'' \simeq \phi - \frac{1}{2}(1 - \frac{n_{rz}}{n_{ry}}) \sin 2\phi \tag{45d}$$

Thus, to the order of $O(\epsilon)$,

$$\cos m\phi' \simeq \cos m\phi + \frac{m}{4}(1 - \frac{n_{rx}}{n_{rz}}) [\cos (m-2)\phi - \cos (m+2)\phi] \tag{45e}$$

$$\sin m\phi'' \simeq \sin m\phi - \frac{m}{4}(1 - \frac{n_{rz}}{n_{ry}}) [\sin (m+2)\phi - \sin (m-2)\phi] \tag{45f}$$

Making use of these approximations in conjunction with Neumann's addition theorem for the Bessel functions [31] and keeping only terms of the order of 1 and ϵ, we have

$$J_m(\frac{n_{rz}}{n_{rx}} U_{rx} \frac{r'}{a}) \simeq J_m(\frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a}) - \frac{1}{4} \frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a}(1 - \frac{n_{rx}^2}{n_{rz}^2})(1 - \cos 2\phi)[J'_m(\frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a})]$$

$$J_m(\frac{n_{rz}}{n_{ry}} U_{ry} \frac{r''}{a}) \simeq J_m(\frac{n_{rz}}{n_{ry}} U_{ry} \frac{r}{a}) - \frac{1}{4} \frac{n_{rz}}{n_{ry}} U_{ry} \frac{r}{a}(1-\frac{n_{ry}^2}{n_{rz}^2})(1+\cos 2\phi)[J'_m(\frac{n_{rz}}{n_{ry}} U_{ry} \frac{r}{a})]$$

where J'_m indicates the derivative of the Bessel functions with respect to its argument.

Use these approximations and sorting out the terms with different harmonics, we have

$$e_x^{(1)} \simeq A_0 [J_0(\frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a}) + \frac{1}{4} \frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a}(1-\frac{n_{rx}^2}{n_{rz}^2})J_1(\frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a})]$$

$$+ A_2 [\frac{1}{8} \frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a}(1-\frac{n_{rx}^2}{n_{rz}^2})J'_2(\frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a}) + \frac{2}{4}(1-\frac{n_{rx}}{n_{rz}})J_2(\frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a})]$$

$$-A_0 \frac{1}{4} \frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a}(1-\frac{n_{rx}^2}{n_{rz}^2})J_1(\frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a}) \cos 2\phi$$

$$+ A_2 [J_2(\frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a}) - \frac{1}{4} \frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a}(1-\frac{n_{rx}^2}{n_{rz}^2})J'_2(\frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a})] \cos 2\phi$$

$$+ A_4 [\frac{1}{8} \frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a}(1-\frac{n_{rx}^2}{n_{rz}^2})J'_4(\frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a}) + \frac{4}{4}(1-\frac{n_{rx}}{n_{rz}})J_4(\frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a})] \cos 2\phi$$

$$+ A_2 [\frac{1}{8} \frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a}(1-\frac{n_{rx}^2}{n_{rz}^2})J'_2(\frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a}) - \frac{2}{4}(1-\frac{n_{rx}}{n_{rz}})J_2(\frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a})] \cos 4\phi$$

$$+ A_4 [J_4(\frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a}) - \frac{1}{4} \frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a}(1-\frac{n_{rx}^2}{n_{rz}^2})J'_4(\frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a})] \cos 4\phi$$

$$+ A_4 [\frac{1}{8} \frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a}(1-\frac{n_{rx}^2}{n_{rz}^2})J'_4(\frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a}) - \frac{4}{4}(1-\frac{n_{rx}}{n_{rz}})J_4(\frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a})] \cos 6\phi$$

$$+\ \frac{1}{4}\ \frac{n_{ry}(n_{rz}^2-n_{ry}^2)}{n_{rz}(n_{rx}^2-n_{ry}^2)}\ \frac{n_{ry}^2-N^2}{N^2}\left[B_2\ J_0(\frac{n_{rz}}{n_{ry}}\ U_{ry}\ \frac{r}{a}) \right.$$

$$+B_4\ J_2(\frac{n_{rz}}{n_{ry}}\ U_{ry}\ \frac{r}{a})\cos 2\phi$$

$$+(B_6-B_2)\ J_4(\frac{n_{rz}}{n_{ry}}\ U_{ry}\ \frac{r}{a})\cos 4\phi$$

$$\left. -\ B_4\ J_6(\frac{n_{rz}}{n_{ry}}\ U_{ry}\ \frac{r}{a})\cos 6\phi \right] + \ldots \tag{46}$$

$$e_y^{(1)} \simeq B_2\ [\ J_2(\frac{n_{rz}}{n_{ry}}\ U_{ry}\ \frac{r}{a}) - \frac{1}{4}\ \frac{n_{rz}}{n_{ry}}\ U_{ry}\ \frac{r}{a}(1-\frac{n_{ry}^2}{n_{rz}^2})J'_2(\frac{n_{rz}}{n_{ry}}\ U_{ry}\ \frac{r}{a})\ ]\ \sin 2\phi$$

$$-\ B_4\ \ [\ \frac{1}{8}\ \frac{n_{rz}}{n_{ry}}\ U_{ry}\ \frac{r}{a}(1-\frac{n_{ry}^2}{n_{rz}^2})J'_4(\frac{n_{rz}}{n_{ry}}\ U_{ry}\ \frac{r}{a}) - \frac{4}{4}(1-\frac{n_{rz}}{n_{ry}})J_4(\frac{n_{rz}}{n_{ry}}\ U_{ry}\ \frac{r}{a})\ ]\ \sin 2\phi$$

$$-B_2\ [\ \frac{1}{8}\ \frac{n_{rz}}{n_{ry}}\ U_{ry}\ \frac{r}{a}(1-\frac{n_{ry}^2}{n_{rz}^2})J'_2(\frac{n_{rz}}{n_{ry}}\ U_{ry}\ \frac{r}{a})\ +\ \frac{2}{4}(1-\frac{n_{rz}}{n_{ry}})J_2(\frac{n_{rz}}{n_{ry}}\ U_{ry}\ \frac{r}{a})\ ]\ \sin 4\phi$$

$$B_4\ [\ J_4(\frac{n_{rz}}{n_{ry}}\ U_{ry}\ \frac{r}{a}) - \frac{1}{4}\ \frac{n_{rz}}{n_{ry}}\ U_{ry}\ \frac{r}{a}(1-\frac{n_{ry}^2}{n_{rz}^2})J'_4(\frac{n_{rz}}{n_{ry}}\ U_{ry}\ \frac{r}{a})\ ]\ \sin 4\phi$$

$$-\ B_6\ \ [\ \frac{1}{8}\ \frac{n_{rz}}{n_{ry}}\ U_{ry}\ \frac{r}{a}(1-\frac{n_{ry}^2}{n_{rz}^2})J'_6(\frac{n_{rz}}{n_{ry}}\ U_{ry}\ \frac{r}{a}) - \frac{6}{4}(1-\frac{n_{rz}}{n_{ry}})J_6(\frac{n_{rz}}{n_{ry}}\ U_{ry}\ \frac{r}{a})\ ]\ \sin 4\phi$$

$$-\ B_4\ \ [\ \frac{1}{8}\ \frac{n_{rz}}{n_{ry}}\ U_{ry}\ \frac{r}{a}(1-\frac{n_{ry}^2}{n_{rz}^2})J'_4(\frac{n_{rz}}{n_{ry}}\ U_{ry}\ \frac{r}{a})\ +\ \frac{4}{4}(1-\frac{n_{rz}}{n_{ry}})J_4(\frac{n_{rz}}{n_{ry}}\ U_{ry}\ \frac{r}{a})\ ]\ \sin 6\phi$$

$$+\ B_6\ [\ J_6(\frac{n_{rz}}{n_{ry}}\ U_{ry}\ \frac{r}{a}) - \frac{1}{4}\ \frac{n_{rz}}{n_{ry}}\ U_{ry}\ \frac{r}{a}(1-\frac{n_{ry}^2}{n_{rz}^2})J'_6(\frac{n_{rz}}{n_{ry}}\ U_{ry}\ \frac{r}{a})\ ]\ \sin 6\phi$$

$$-B_6 \left[\frac{1}{8} \frac{n_{rz}}{n_{ry}} U_{ry} \frac{r}{a}(1-\frac{n_{ry}^2}{n_{rz}^2})J'_6(\frac{n_{rz}}{n_{ry}} U_{ry} \frac{r}{a}) + \frac{6}{4}(1-\frac{n_{rz}}{n_{ry}})J_6(\frac{n_{rz}}{n_{ry}} U_{ry} \frac{r}{a}) \right] \sin 8\phi$$

$$+ \frac{1}{4} \frac{n_{rx}(n_{rz}^2-n_{rx}^2)}{n_{rz}(n_{ry}^2-n_{rx}^2)} \frac{n_{rx}^2-N^2}{N^2} \Bigg[[2A_0 - A_4]\, J_2(\frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a}) \sin 2\phi$$

$$+ A_2 \left[J_4(\frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a}) \sin 4\phi \right]$$

$$+ A_4 \left[J_6(\frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a}) \sin 6\phi \right] \Bigg] + \ldots \qquad (47)$$

By rearranging the terms in (46) and (47) in the forms of (21a) and (21b), we have the Fourier coefficients for the core region,

$$e_{x0}^{R}(r) = A_0 f_{00}(r) + A_2 f_{02}(r) + B_2 g_{02}(r) \qquad (48a)$$

$$e_{c2}^{Rc}(r) = A_0 f_{20}(r) + A_2 f_{22}(r) + A_4 f_{24}(r) + B_4 g_{24}(r) \qquad (48b)$$

$$e_{x4}^{Rc}(r) = A_2 f_{42}(r) + A_4 f_{44}(r) + B_2 g_{42}(r) + B_6 g_{46}(r) \qquad (48c)$$

$$e_{y2}^{Rs}(r) = B_2 p_{22}(r) + B_4 p_{24}(r) + A_0 q_{20}(r) + A_4 q_{24}(r) \qquad (48d)$$

$$e_{y4}^{Rs}(r) = B_2 p_{42}(r) + B_4 p_{44}(r) + B_6 p_{46}(r) + A_2 q_{42}(r) \qquad (48e)$$

Explicit expressions of $f_{00}(r)$ etc. are given in Appendix A.

Similarly, from (41) - (44), we have for the cladding region:

$$e_{x0}^{D}(r) = C_0\, h_{00}(r) + C_2\, h_{02}(r) + D_2\, k_{02}(r) \qquad (49a)$$

$$e_{x2}^{Dc}(r) = C_0\, h_{20}(r) + C_2\, h_{22}(r) + C_4\, h_{24}(r) + D_4\, k_{24}(r) \qquad (49b)$$

$$e_{x4}^{Dc}(r) = C_2\, h_{42}(r) + C_4\, h_{44}(r) + D_2\, k_{42}(r) + D_6\, k_{46}(r) \qquad (49c)$$

$$e_{y2}^{Ds}(r) = D_2\, r_{22}(r) + D_4\, r_{24}(r) + C_0\, s_{20}(r) + C_4\, s_{24}(r) \tag{49d}$$

$$e_{y4}^{Ds}(r) = D_2\, r_{42}(r) + D_4\, r_{44}(r) + D_6\, r_{46}(r) + C_2\, s_{42}(r) \tag{49e}$$

Expressions for h_{00} etc. are also listed in Appendix A.

Dispersion relation:

Three leading harmonic terms of the even cosine series of $\mathbf{e}_x$ and the even sine series of $\mathbf{e}_y$ are kept in (41), 42), (50) and (51). The coefficients A_0, A_2, A_4, B_2, B_4 and B_6 are pertaining to the core region and C_0, C_2, C_4, D_2, D_4 and D_6 to the cladding region. Substituting these equations into the boundary condition equations (32a), (32c), (32e), (32g) and (33a), (33c), (33e) and (33g) with j = 3 and 5. we have 12 linear equations which can be cast in matrix form:

$$\begin{bmatrix} Q_{11} & Q_{12} & \cdots & Q_{19} & Q_{1a} & Q_{1b} & Q_{1c} \\ Q_{21} & Q_{22} & \cdots & Q_{29} & Q_{2a} & Q_{2b} & Q_{2c} \\ Q_{31} & Q_{32} & \cdots & Q_{39} & Q_{3a} & Q_{3b} & Q_{3c} \\ Q_{41} & Q_{42} & \cdots & Q_{49} & Q_{4a} & Q_{4b} & Q_{4c} \\ Q_{51} & Q_{52} & \cdots & Q_{59} & Q_{5a} & Q_{5b} & Q_{5c} \\ Q_{61} & Q_{62} & \cdots & Q_{69} & Q_{6a} & Q_{6b} & Q_{6c} \\ Q_{71} & Q_{72} & \cdots & Q_{79} & Q_{7a} & Q_{7b} & Q_{7c} \\ Q_{81} & Q_{82} & \cdots & Q_{89} & Q_{8a} & Q_{8b} & Q_{8c} \\ Q_{91} & Q_{92} & \cdots & Q_{99} & Q_{9a} & Q_{9b} & Q_{9c} \\ Q_{a1} & Q_{a2} & \cdots & Q_{a9} & Q_{aa} & Q_{ab} & Q_{ac} \\ Q_{b1} & Q_{b2} & \cdots & Q_{b9} & Q_{ba} & Q_{bb} & Q_{bc} \\ Q_{c1} & Q_{c2} & \cdots & Q_{c9} & Q_{ca} & Q_{cb} & Q_{cc} \end{bmatrix} \begin{bmatrix} A_0 \\ A_2 \\ A_4 \\ B_2 \\ B_4 \\ B_6 \\ C_0 \\ C_2 \\ C_4 \\ D_2 \\ D_4 \\ D_6 \end{bmatrix} = 0 \tag{50}$$

Expressions for Q_{ij} with i or j = 1, 2, 3,..., 9, 10,a, b, c. are given in Appendix B.

A dispersion relation is obtained by setting the 12×12 determinant to zero.

$$\| Q \| = 0 \tag{51}$$

Obviously, (51) will be solved numerically.

5. Discusssio and Numerical Examples

Before numerical results are presented, it would be helpful to enumerate the similarities and differences between elliptical fibers and biaxial fibers purely from a mathematical view point. As discussed in Section 4, (20a) and (20b) can be reduced to the usual wave equations by a simple substitution: $x'' = (n_{ry}/n_{rz})\,x$ or $y' = (n_{rx}/n_{rz})\,y$ for the core region, and corresponding substitution for the cladding region. Thus a biaxial fiber with a circular cross section can be viewed as an isotropic fiber with an elliptical cross section so far as the differential equations for the zero-order solution are concerned. However the scaling factors for the core and the cladding regions are not necessarily the same, so the ellipticity of the core may differ from that of the cladding. The main difference between two types of fibers is the way $\mathbf{e}_x$ and $\mathbf{e}_y$ are coupled. For elliptical fibers, the coupling occurs only at the core-cladding boundary. For biaxial fibers, coupling originates not only from the boundary, but also from the entire biaxial region or regions. This is due to the terms $\dfrac{\partial^2 \mathbf{e}_x}{\partial x \partial y}$ and $\dfrac{\partial^2 \mathbf{e}_y}{\partial x \partial y}$ in (20a) and (20b). Even for the coupling at the boundary, there is a difference. In matching $\mathbf{d}_r$ at the core/cladding boundary, it is necessary to account for the difference between n_{xr} and n_{yr}, n_{xd} and n_{yr}.

Since the characteristics of optical waveguides depend more on the index differential and/or index anisotropy than the index itself, the results can best be described in terms of index differential and anisotropy. As stated previously, the indices of the core and cladding regions are (n_{rx}, n_{ry}, n_{rz}) and (n_{dx}, n_{dy}, n_{dz}) respectively (Figure 1). The index difference between the core and the cladding regions is defined as

$$d_{rd} = (n_{rm} - n_{dm})/n_{rm} \tag{52}$$

where

n_{rm} = max of (n_{rx}, n_{ry})

n_{dm} = max of (n_{dx}, n_{dy})

Similarly the index anisotropy of the core and the cladding regions are:

$$d_{xyr} = (n_{rx} - n_{ry})/n_{rx} \tag{53a}$$

$$d_{xzr} = (n_{rx} - n_{rz})/n_{rx} \tag{53b}$$

$$d_{xyd} = (n_{dx} - n_{dy})/n_{dx} \tag{53c}$$

$$d_{xzd} = (n_{dx} - n_{dz})/n_{dx} \tag{53d}$$

In all numerical calculations, n_{xr} is set to 1.5000 while other indices are varied by changing d_{xyr}, d_{xyd} and d_{rd} etc. To make the numerical results more meaningful, they are cast in terms of generalized parameters V and b,

$$V = ka\sqrt{n_{rm}^2 - n_{dm}^2} \tag{54a}$$

$$b = (N^2 - n_{dm}^2) / (n_{rm}^2 - n_{dm}^2) \tag{54b}$$

which are the same as those used for isotropic fibers.

The transverse electric fields for two lowest polarization modes have been obtained, using the method developed in the previous sections. The fields for the lowest polarization modes are depicted in Figures 2a and 2b, and those for the second lowest polarization mode are given in Figures 3a and 3b. In these plots, $|\mathbf{e}_x|$ and $|\mathbf{e}_y|$ are plotted as functions of x/a and y/a. To reveal the details of each field components clearly, it is necessary to use different scale factors for different plots. To show the relative amplitude of $|\mathbf{e}_x|$ and $|\mathbf{e}_y|$, the ratio of $|\mathbf{e}_x|_{peak}/|\mathbf{e}_y|_{peak}$ or $|\mathbf{e}_y|_{peak}/|\mathbf{e}_x|_{peak}$ is also indicated in the plots. For the lowest order polarization mode (Figures 2a and 2b), the peak of $|\mathbf{e}_x|$ is approximately 128 times stronger than that of

$|\mathbf{e}_y|$. On the other hand, for the second lowest order polarization mode of the same fiber and with the same V value (Figures 3a and 3b), $|\mathbf{e}_y|$ is much stronger than that of $|\mathbf{e}_x|$. Comparison of these plots also reveal that that the $|\mathbf{e}_x|$ and $|\mathbf{e}_y|$ of the lowest polarization mode are very similar to that of $|\mathbf{e}_y|$ and $|\mathbf{e}_x|$ of the second lowest order polarization mode. Similar results have also been obtained for the same fiber with different V value or for different fibers. In summary, we note for each polarization mode, both field components are present, one field component is much stronger than the other. In addition, the dominant field component ($|\mathbf{e}_x|$ of Figure 2a and $|\mathbf{e}_y|$ of Figure 3b) has a peak at the center of the fiber, just like those of the LP_{11} modes for the conventional isotropic fibers. The accompanying and much weaker field component (i.e. $|\mathbf{e}_y|$ of Figure 2b and $|\mathbf{e}_x|$ of Figure 3a respectively) has an entirely different look: a null at the fiber center, surrounded by four peaks, one in each quadrant. Our calculations also show that fields extend much more into the cladding region when the V value is small, i.e., the fibers are operating near cutoff. Of course this is expected.

For the fiber discussed, $n_{rx} > n_{ry}$ and $n_{dx} > n_{dy}$. and the dominant electric field of the lowest polarization mode is $\mathbf{e}_x$. On the other hand, for the second lowest order polarization mode, the dominant electric field is $\mathbf{e}_y$. In general, for a given fiber, the lowest order polarization mode is the mode with the dominant electric fields along the direction with the large index of refraction. Since this polarization mode has the largest propagation constant, it will be identified as mode s. The second lowest order polarization mode, labeled as mode f, has its dominant electric field along a direction perpendicular to that of mode s. The normalized b parameters, as defined in (54a) and (54b), of these two polarization modes are labeled as b_s and b_f. Curves of b_s vs V for various values of d_{xyr} and d_{xyd} etc. are indistinguishable, as shown in Figure 4. However, curves for b_f do vary with d_{xyr} and d_{xyd}. Of particular interest is the $b_s - b_f$. In

Figures 5, 6 and 7, $b_s - b_f$ is plotted as a function of V with d_{xyr} and d_{xyd} as parameters. In Figure 5, d_{xyr} is kept to a constant while d_{xyd} varies. We note that near cutoff, $b_s - b_f$ is appreciable if d_{xyd} is large. This is understandable. When a fiber is operating near cutoff, fields spread out to the cladding region, and therefore, the cladding anisotropy d_{xyd} should and do have an strong influence on the fiber characteristics. On the other hand, for fibers operating far from cutoff, fields are concentrated in the core region and therefore d_{xyr} has a large impact on the $b_s - b_f$. This is shown in Figure 6 where d_{xyd} are kept constant while d_{xyr} varies. In Figure 7, $d_{xyr} = d_{xyd}$ and they vary simultaneously. For these cases $b_s - b_f$ is practically independent of V.

6. Coupling with Biaxial Fibers

Although the excitation of low-birefringent fibers by plane waves or gaussian beams, or the coupling between these fibers with planar or channel waveguides have been studied extensively, the corresponding problems involving high-birefringence fibers have not been treated previously. The main stumbling block is the lack of an accurate description of fields in high birefringence fibers. With the theory developed in this report, such a difficulty no longer exists and one shall be able to proceed to study these coupling problems. In the following, we shall outline the basic procedure for treating the coupling problems and present some of the results.

Let the transverse field components of the polarization mode ν be $\vec{e}_{t\nu}$ and $\vec{h}_{t\nu}$. Then the transverse electric fields in the fiber can be expressed as

$$\vec{e}_t = \sum c_\nu \vec{e}_{t\nu}. \tag{55}$$

For a given incident field $\vec{E}_{in}$, c_ν is given by

$$c_\nu = \int (\vec{E}^{in} \times \vec{h}^*_{t\nu}) \cdot \hat{a}_z \, ds \tag{56}$$

where $\hat{a}_z$ is the unit vector in the z direction. Thus the dependence of c_ν as a function

of the state of polarization (SOP) of the incident field and under various excitation conditions can be determined. We assume that $\vec{e}_{t\nu}$ and $\vec{h}_{t\nu}$ are normalized so that the power coupled to the polarization mode ν is proportional to $|c_\nu|^2$.

This approach has been employed to study the excitation of low birefringent as well as high birefringent fibers. The contrast between these two types of fibers is quite interesting. Take the case of plane wave excitation as an example. As the input SOP of changes, the SOP of the fields in the low-birefringent fibers changes with the input SOP, but the total power coupled into the fibers remains unchanged. On the other hand, for high-birefringence fibers, both the SOP and the total power vary as the input SOP varies.

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Appendix A

In this Appendix, the expressions of $f_{00}(r)$ etc. are listed.

I. Core region:

$$f_{00}(r) = J_0(\frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a}) + \frac{1}{4} \frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a}(1-\frac{n_{rx}^2}{n_{rz}^2})J_1(\frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a})$$

$$f_{02}(r) = \frac{1}{8} \frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a}(1-\frac{n_{rx}^2}{n_{rz}^2})J'_2(\frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a}) + \frac{2}{4}(1-\frac{n_{rx}}{n_{rz}})J_2(\frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a})$$

$$f_{20}(r) = -\frac{1}{4} \frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a}(1-\frac{n_{rx}^2}{n_{rz}^2})J_1(\frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a})$$

$$f_{22}(r) = J_2(\frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a}) - \frac{1}{4} \frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a}(1-\frac{n_{rx}^2}{n_{rz}^2})J'_2(\frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a})$$

$$f_{24}(r) = \frac{1}{8} \frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a}(1-\frac{n_{rx}^2}{n_{rz}^2})J'_4(\frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a}) + \frac{4}{4}(1-\frac{n_{rx}}{n_{rz}})J_4(\frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a})$$

$$f_{42}(r) = \frac{1}{8} \frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a}(1-\frac{n_{rx}^2}{n_{rz}^2})J'_2(\frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a}) - \frac{2}{4}(1-\frac{n_{rx}}{n_{rz}})J_2(\frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a})$$

$$f_{44}(r) = J_4(\frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a}) - \frac{1}{4} \frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a}(1-\frac{n_{rx}^2}{n_{rz}^2})J'_4(\frac{n_{rz}}{n_{rx}} U_{rx} \frac{r}{a})$$

$$g_{02}(r) = \frac{1}{4} \frac{n_{ry}(n_{rz}^2-n_{ry}^2)}{n_{rz}(n_{rx}^2-n_{ry}^2)} \frac{n_{ry}^2-N^2}{N^2} J_0(\frac{n_{rz}}{n_{ry}} U_{ry} \frac{r}{a})$$

$$g_{24}(r) = \frac{1}{4} \frac{n_{ry}(n_{rz}^2-n_{ry}^2)}{n_{rz}(n_{rx}^2-n_{ry}^2)} \frac{n_{ry}^2-N^2}{N^2} J_2(\frac{n_{rz}}{n_{ry}} U_{ry} \frac{r}{a})$$

$$g_{42}(r) = -\frac{1}{4}\ \frac{n_{ry}(n_{rz}^2-n_{ry}^2)}{n_{rz}(n_{rx}^2-n_{ry}^2)}\ \frac{n_{ry}^2-N^2}{N^2} J_4(\frac{n_{rz}}{n_{ry}}\ U_{ry}\ \frac{r}{a})$$

$$g_{46}(r) = +\frac{1}{4}\ \frac{n_{ry}(n_{rz}^2-n_{ry}^2)}{n_{rz}(n_{rx}^2-n_{ry}^2)}\ \frac{n_{ry}^2-N^2}{N^2} J_4(\frac{n_{rz}}{n_{ry}}\ U_{ry}\ \frac{r}{a})$$

$$p_{22}(r) = J_2(\frac{n_{rz}}{n_{ry}}\ U_{ry}\ \frac{r}{a}) - \frac{1}{4}\ \frac{n_{rz}}{n_{ry}}\ U_{ry}\ \frac{r}{a}(1-\frac{n_{ry}^2}{n_{rz}^2})J'_2(\frac{n_{rz}}{n_{ry}}\ U_{ry}\ \frac{r}{a})$$

$$p_{24}(r) = -\frac{1}{8}\ \frac{n_{rz}}{n_{ry}}\ U_{ry}\ \frac{r}{a}(1-\frac{n_{ry}^2}{n_{rz}^2})J'_4(\frac{n_{rz}}{n_{ry}}\ U_{ry}\ \frac{r}{a}) - \frac{4}{4}(1-\frac{n_{rz}}{n_{ry}})J_4(\frac{n_{rz}}{n_{ry}}\ U_{ry}\ \frac{r}{a})$$

$$p_{42}(r) = -\frac{1}{8}\ \frac{n_{rz}}{n_{ry}}\ U_{ry}\ \frac{r}{a}(1-\frac{n_{ry}^2}{n_{rz}^2})J'_2(\frac{n_{rz}}{n_{ry}}\ U_{ry}\ \frac{r}{a}) + \frac{2}{4}(1-\frac{n_{rz}}{n_{ry}})J_2(\frac{n_{rz}}{n_{ry}}\ U_{ry}\ \frac{r}{a})$$

$$p_{44}(r) = J_4(\frac{n_{rz}}{n_{ry}}\ U_{ry}\ \frac{r}{a}) - \frac{1}{4}\ \frac{n_{rz}}{n_{ry}}\ U_{ry}\ \frac{r}{a}(1-\frac{n_{ry}^2}{n_{rz}^2})J'_4(\frac{n_{rz}}{n_{ry}}\ U_{ry}\ \frac{r}{a})$$

$$p_{46}(r) = -\frac{1}{8}\ \frac{n_{rz}}{n_{ry}}\ U_{ry}\ \frac{r}{a}(1-\frac{n_{ry}^2}{n_{rz}^2})J'_6(\frac{n_{rz}}{n_{ry}}\ U_{ry}\ \frac{r}{a}) - \frac{6}{4}(1-\frac{n_{rz}}{n_{ry}})J_6(\frac{n_{rz}}{n_{ry}}\ U_{ry}\ \frac{r}{a})$$

$$q_{20}(r) = \frac{1}{2}\ \frac{n_{rx}(n_{rz}^2-n_{rx}^2)}{n_{rz}(n_{ry}^2-n_{rx}^2)}\ \frac{n_{rx}^2-N^2}{N^2}\ J_2(\frac{n_{rz}}{n_{rx}}\ U_{rx}\ \frac{r}{a})$$

$$q_{24}(r) = -\frac{1}{4}\ \frac{n_{rx}(n_{rz}^2-n_{rx}^2)}{n_{rz}(n_{ry}^2-n_{rx}^2)}\ \frac{n_{rx}^2-N^2}{N^2}\ J_2(\frac{n_{rz}}{n_{rx}}\ U_{rx}\ \frac{r}{a})$$

$$q_{42}(r) = +\frac{1}{4}\ \frac{n_{rx}(n_{rz}^2-n_{rx}^2)}{n_{rz}(n_{ry}^2-n_{rx}^2)}\ \frac{n_{rx}^2-N^2}{N^2} J_4(\frac{n_{rz}}{n_{rx}}\ U_{rx}\ \frac{r}{a})$$

II. Cladding region:

$$h_{00}(r) = [K_0(\frac{n_{dz}}{n_{dx}} W_{dx} \frac{r}{a}) + \frac{1}{4} \frac{n_{dz}}{n_{dx}} W_{dx} \frac{r}{a}(1-\frac{n_{dx}^2}{n_{dz}^2})K_1(\frac{n_{dz}}{n_{dx}} W_{dx} \frac{r}{a})]$$

$$h_{20}(r) = -\frac{1}{4} \frac{n_{dz}}{n_{dx}} W_{dx} \frac{r}{a}(1-\frac{n_{dx}^2}{n_{dz}^2})K_1(\frac{n_{dz}}{n_{dx}} W_{dx} \frac{r}{a})$$

$$h_{22}(r) = [K_2(\frac{n_{dz}}{n_{dx}} W_{dx} \frac{r}{a}) - \frac{1}{4} \frac{n_{dz}}{n_{dx}} W_{dx} \frac{r}{a}(1-\frac{n_{dx}^2}{n_{dz}^2})K'_2(\frac{n_{dz}}{n_{dx}} W_{dx} \frac{r}{a})]$$

$$h_{02}(r) = +[\frac{1}{8} \frac{n_{dz}}{n_{dx}} W_{dx} \frac{r}{a}(1-\frac{n_{dx}^2}{n_{dz}^2})K'_2(\frac{n_{dz}}{n_{dx}} W_{dx} \frac{r}{a}) + \frac{2}{4}(1-\frac{n_{dx}}{n_{dz}})K_2(\frac{n_{dz}}{n_{dx}} W_{dx} \frac{r}{a})]$$

$$h_{42}(r) = +[\frac{1}{8} \frac{n_{dz}}{n_{dx}} W_{dx} \frac{r}{a}(1-\frac{n_{dx}^2}{n_{dz}^2})K'_2(\frac{n_{dz}}{n_{dx}} W_{dx} \frac{r}{a}) - \frac{2}{4}(1-\frac{n_{dx}}{n_{dz}})K_2(\frac{n_{dz}}{n_{dx}} W_{dx} \frac{r}{a})]$$

$$h_{44}(r) = [K_4(\frac{n_{dz}}{n_{dx}} W_{dx} \frac{r}{a}) - \frac{1}{4} \frac{n_{dz}}{n_{dx}} W_{dx} \frac{r}{a}(1-\frac{n_{dx}^2}{n_{dz}^2})K'_4(\frac{n_{dz}}{n_{dx}} W_{dx} \frac{r}{a})]$$

$$h_{24}(r) = +[\frac{1}{8} \frac{n_{dz}}{n_{dx}} W_{dx} \frac{r}{a}(1-\frac{n_{dx}^2}{n_{dz}^2})K'_4(\frac{n_{dz}}{n_{dx}} W_{dx} \frac{r}{a}) + \frac{4}{4}(1-\frac{n_{dx}}{n_{dz}})K_4(\frac{n_{dz}}{n_{dx}} W_{dx} \frac{r}{a})]$$

$$h_{46}(r) = +[\frac{1}{8} \frac{n_{dz}}{n_{dx}} W_{dx} \frac{r}{a}(1-\frac{n_{dx}^2}{n_{dz}^2})K'_6(\frac{n_{dz}}{n_{dx}} W_{dx} \frac{r}{a}) + \frac{6}{4}(1-\frac{n_{dx}}{n_{dz}})K_6(\frac{n_{dz}}{n_{dx}} W_{dx} \frac{r}{a})]$$

$$r_{22}(r) = [K_2(\frac{n_{dz}}{n_{dy}} W_{dy} \frac{r}{a}) - \frac{1}{4} \frac{n_{dz}}{n_{dy}} W_{dy} \frac{r}{a}(1-\frac{n_{dy}^2}{n_{dz}^2})K'_2(\frac{n_{dz}}{n_{dy}} W_{dy} \frac{r}{a})]$$

$$r_{42}(r) = -[\frac{1}{8} \frac{n_{dz}}{n_{dy}} W_{dy} \frac{r}{a}(1-\frac{n_{dy}^2}{n_{dz}^2})K'_2(\frac{n_{dz}}{n_{dy}} W_{dy} \frac{r}{a}) + \frac{2}{4}(1-\frac{n_{dz}}{n_{dy}})K_2(\frac{n_{dz}}{n_{dy}} W_{dy} \frac{r}{a})]$$

$$r_{44}(r) = [K_4(\frac{n_{dz}}{n_{dy}} W_{dy} \frac{r}{a}) - \frac{1}{4} \frac{n_{dz}}{n_{dy}} W_{dy} \frac{r}{a}(1-\frac{n_{dy}^2}{n_{dz}^2})K'_4(\frac{n_{dz}}{n_{dy}} W_{dy} \frac{r}{a})]$$

$$r_{24}(r) = -[\frac{1}{8}\ \frac{n_{dz}}{n_{dy}}\ W_{dy}\ \frac{r}{a}(1-\frac{n_{dy}^2}{n_{dz}^2})K'_4(\frac{n_{dz}}{n_{dy}}\ W_{dy}\ \frac{r}{a}) - \frac{4}{4}(1-\frac{n_{dz}}{n_{dy}})K_4(\frac{n_{dz}}{n_{dy}}\ W_{dy}\ \frac{r}{a})]$$

$$r_{46}(r) = -[\frac{1}{8}\ \frac{n_{dz}}{n_{dy}}\ W_{dy}\ \frac{r}{a}(1-\frac{n_{dy}^2}{n_{dz}^2})K'_6(\frac{n_{dz}}{n_{dy}}\ W_{dy}\ \frac{r}{a}) - \frac{6}{4}(1-\frac{n_{dz}}{n_{dy}})K_6(\frac{n_{dz}}{n_{dy}}\ W_{dy}\ \frac{r}{a})]$$

$$k_{02}(r) = -\ \frac{1}{4}\ \frac{n_{dy}(n_{dz}^2-n_{dy}^2)}{n_{dz}(n_{dx}^2-n_{dy}^2)}\ \frac{n_{dy}^2-N^2}{N^2}[\ K_0(\frac{n_{dz}}{n_{dy}}\ W_{dy}\ \frac{r}{a})]$$

$$k_{42}(r) = +\ \frac{1}{4}\ \frac{n_{dy}(n_{dz}^2-n_{dy}^2)}{n_{dz}(n_{dx}^2-n_{dy}^2)}\ \frac{n_{dy}^2-N^2}{N^2}[\ K_4(\frac{n_{dz}}{n_{dy}}\ W_{dy}\ \frac{r}{a})]$$

$$k_{24}(r) = -\ \frac{1}{4}\ \frac{n_{dy}(n_{dz}^2-n_{dy}^2)}{n_{dz}(n_{dx}^2-n_{dy}^2)}\ \frac{n_{dy}^2-N^2}{N^2}[\ K_2(\frac{n_{dz}}{n_{dy}}\ W_{dy}\ \frac{r}{a})]$$

$$k_{46}(r) = -\ \frac{1}{4}\ \frac{n_{dy}(n_{dz}^2-n_{dy}^2)}{n_{dz}(n_{dx}^2-n_{dy}^2)}\ \frac{n_{dy}^2-N^2}{N^2}[\ K_4(\frac{n_{dz}}{n_{dy}}\ W_{dy}\ \frac{r}{a})]$$

$$s_{20}(r) = -\ 2\ \frac{1}{4}\ \frac{n_{dx}(n_{dz}^2-n_{dx}^2)}{n_{dz}(n_{dy}^2-n_{dx}^2)}\ \frac{n_{dx}^2-N^2}{N^2}\ K_2(\frac{n_{dz}}{n_{dx}}\ W_{dx}\ \frac{r}{a})$$

$$s_{42}(r) = -\ \frac{1}{4}\ \frac{n_{dx}(n_{dz}^2-n_{dx}^2)}{n_{dz}(n_{dy}^2-n_{dx}^2)}\ \frac{n_{dx}^2-N^2}{N^2}\quad [K_4(\frac{n_{dz}}{n_{dx}}\ W_{dx}\ \frac{r}{a})]$$

$$s_{24}(r) = +\ \frac{1}{4}\ \frac{n_{dx}(n_{dz}^2-n_{dx}^2)}{n_{dz}(n_{dy}^2-n_{dx}^2)}\ \frac{n_{dx}^2-N^2}{N^2}\quad [\ K_2(\frac{n_{dz}}{n_{dx}}\ W_{dx}\ \frac{r}{a})]$$

$$s_{46}(r) = +\ \frac{1}{4}\ \frac{n_{dx}(n_{dz}^2-n_{dx}^2)}{n_{dz}(n_{dy}^2-n_{dx}^2)}\ \frac{n_{dx}^2-N^2}{N^2}\quad [\ K_4(\frac{n_{dz}}{n_{dx}}\ W_{dx}\ \frac{r}{a})]$$

Appendix B

The expressions for the matrix elements of Q matrix are given in this Appendix. Functions $f_{00}(r)$ etc. are listed in Appendix A. A ′ signifies differentiation with respect to r.

$$Q_{1,1} = f_{20}(r) - 2\, f_{00}(r) + q_{20}(r)$$

$$Q_{1,2} = f_{22}(r) - 2\, f_{02}(r)$$

$$Q_{1,3} = f_{24}(r) + q_{24}(r)$$

$$Q_{1,4} = -2\, g_{02}(r) + p_{22}(r)$$

$$Q_{1,5} = g_{24}(r) + p_{24}(r)$$

$$Q_{1,6} = 0$$

$$Q_{1,7} = -\left[h_{20}(r) - 2\, h_{00}(r) + s_{20}(r)\right]$$

$$Q_{1,8} = -\left[h_{22}(r) - 2\, h_{02}(r)\right]$$

$$Q_{1,9} = -\left[h_{24}(r) + s_{24}(r)\right]$$

$$Q_{1,a} = 2\, g_{02}(r) - r_{22}(r)$$

$$Q_{1,b} = -\left[r_{24}(r) + g_{24}(r)\right]$$

$$Q_{1,c} = 0$$

$$Q_{2,1} = \frac{n_{rx}^2}{n_{rz}^2}\left[2\, f'_{00}(r) + f'_{20}(r) + 2\; f_{20}(r)\right] + \frac{n_{ry}^2}{n_{rz}^2}\left[q'_{20}(r) + 2\; q_{20}(r)\right]$$

$$Q_{2,2} = \frac{n_{rx}^2}{n_{rz}^2} [2\, f'_{02}(r) + f'_{22}(r) + 2\ f_{22}(r)]$$

$$Q_{2,3} = \frac{n_{rx}^2}{n_{rz}^2} [f'_{24}(r) + 2\ f_{24}(r)] + \frac{n_{ry}^2}{n_{rz}^2} [q'_{24}(r) + 2\ q_{24}(r)]$$

$$Q_{2,4} = \frac{n_{rx}^2}{n_{rz}^2} g'_{02}(r) + \frac{n_{ry}^2}{n_{rz}^2} [p'_{22}(r) + 2\ p_{22}(r)]$$

$$Q_{2,5} = \frac{n_{rx}^2}{n_{rz}^2} [g'_{24}(r) + 2\ g_{24}(r)] + \frac{n_{ry}^2}{n_{rz}^2} [p'_{24}(r) + 2\ p_{24}(r)]$$

$$Q_{2,6} = 0$$

$$Q_{2,7} = -\frac{n_{dx}^2}{n_{dz}^2} [2\, h'_{00}(r) + h'_{20}(r) + 2\ h_{20}(r)] - \frac{n_{dy}^2}{n_{dz}^2} [s'_{20}(r) + 2\ s_{20}(r)]$$

$$Q_{2,8} = -\frac{n_{dx}^2}{n_{dz}^2} [2\, h'_{20}(r) + h'_{22}(r) + 2\ h_{22}(r)]$$

$$Q_{2,9} = -\frac{n_{dx}^2}{n_{dz}^2} [h'_{24}(r) + 2\ h_{24}(r)] - \frac{n_{dy}^2}{n_{dz}^2} [s'_{24}(r) + 2\ s_{24}(r)]$$

$$Q_{2,a} = -\frac{n_{dx}^2}{n_{dz}^2} k'_{02}(r) - \frac{n_{dy}^2}{n_{dz}^2} [r'_{22}(r) + 2\ r_{22}(r)]$$

$$Q_{2,b} = -\frac{n_{dx}^2}{n_{dz}^2} [k'_{24}(r) + 2\ g_{24}(r)] - \frac{n_{dy}^2}{n_{dz}^2} [r'_{24}(r) + 2\ r_{24}(r)]$$

$$Q_{2,c} = 0$$

$$Q_{3,1} = n_{rx}^2 [f_{20}(r) + 2\, f_{00}(r)] + n_{ry}^2\, q_{20}(r)$$

$$Q_{3,2} = n_{rx}^2 \, [f_{22}(r) + 2 \, f_{02}(r)]$$

$$Q_{3,3} = n_{rx}^2 \, f_{24}(r) + n_{ry}^2 \, q_{24}(r)$$

$$Q_{3,4} = n_{rx}^2 \, 2 \, g_{02}(r) + n_{ry}^2 \, p_{22}(r)$$

$$Q_{3,5} = n_{rx}^2 \, g_{24}(r) + n_{ry}^2 \, p_{24}(r)$$

$$Q_{3,6} = 0$$

$$Q_{3,7} = - n_{dx}^2 \, [h_{20}(r) + 2 \, h_{00}(r)] - n_{dy}^2 \, s_{20}(r)$$

$$Q_{3,8} = - n_{dx}^2 \, [h_{22}(r) + 2 \, h_{02}(r)]$$

$$Q_{3,9} = - n_{dx}^2 \, h_{24}(r) - n_{dy}^2 \, s_{24}(r)$$

$$Q_{3,a} = - n_{dx}^2 \, 2 \, g_{02}(r) - n_{dy}^2 \, r_{22}(r)$$

$$Q_{3,b} = - n_{dx}^2 \, g_{24}(r) - n_{dy}^2 \, r_{24}(r)$$

$$Q_{3,c} = 0$$

$$Q_{4,1} = f'_{20}(r) - 2 \, f'_{00}(r) + 2 \, f_{20}(r) + 2 \, q_{20}(r) + q'_{20}(r)$$

$$Q_{4,2} = f'_{22}(r) + 2 \, f_{22}(r) - 2 \, f'_{02}(r)$$

$$Q_{4,3} = f'_{24}(r) + q'_{24}(r) + 2 \, f_{24}(r) + 2 \, q_{24}(r)$$

$$Q_{4,4} = - 2 \, g'_{02}(r) + p'_{22}(r) + 2 \, p_{22}(r)$$

$$Q_{4,5} = g'_{24}(r) + p'_{24}(r) + 2 \, g_{24}(r) + 2 \, p_{24}(r)$$

$$Q_{4,6} = 0$$

$$Q_{4,7} = -[h'_{20}(r) - 2\,h'_{00}(r) + 2\;h_{20}(r) + 2\;s_{20}(r) + s'_{20}(r)]$$

$$Q_{4,8} = -[h'_{22}(r) + 2\;h_{22}(r) - 2\,h'_{02}(r)]$$

$$Q_{4,9} = -[h'_{24}(r) + s'_{24}(r) + 2\;h_{24}(r) + 2\;s_{24}(r)]$$

$$Q_{4,a} = 2\,k'_{02}(r) - r'_{22}(r) - 2\;r_{22}(r)$$

$$Q_{4,b} = -[k'_{24}(r) + r'_{24}(r) + 2\;g_{24}(r) + 2\;r_{24}(r)]$$

$$Q_{4,c} = 0$$

$$Q_{5,1} = -f_{20}(r) + q_{20}(r)$$

$$Q_{5,2} = -f_{22}(r) + f_{42}(r) + q_{42}(r)$$

$$Q_{5,3} = -f_{24}(r) + f_{44}(r) + q_{24}(r)$$

$$Q_{5,4} = g_{42}(r) + p_{22}(r) + p_{42}(r)$$

$$Q_{5,5} = -g_{24}(r) + p_{24}(r) + p_{44}(r)$$

$$Q_{5,6} = g_{46}(r) + p_{46}(r)$$

$$Q_{5,7} = h_{20}(r) - s_{20}(r)$$

$$Q_{5,8} = h_{22}(r) - h_{42}(r) - s_{42}(r)$$

$$Q_{5,9} = h_{24}(r) - h_{44}(r) - s_{24}(r)$$

$$Q_{5,a} = -[g_{42}(r) + r_{22}(r) + r_{42}(r)]$$

$$Q_{5,b} = g_{24}(r) - r_{24}(r) - r_{44}(r)$$

$$Q_{5,c} = -[g_{46}(r) + r_{46}(r)]$$

$$Q_{6,1} = \frac{n_{rx}^2}{n_{rz}^2}[f'_{20}(r) - 2\ f_{20}(r)] + \frac{n_{ry}^2}{n_{rz}^2}[-q'_{20}(r) + 2\ q_{20}(r)]$$

$$Q_{6,2} = \frac{n_{rx}^2}{n_{rz}^2}[f'_{22}(r) + f'_{42}(r) + 4\ f_{42}(r) - 2\ f_{22}(r)] + \frac{n_{ry}^2}{n_{rz}^2}[q'_{42}(r) + 4\ q_{42}(r)]$$

$$Q_{6,3} = \frac{n_{rx}^2}{n_{rz}^2}[f'_{24}(r) + f'_{44}(r) + 4\ f_{44}(r) - 2\ f_{24}(r)] + \frac{n_{ry}^2}{n_{rz}^2}[-q'_{24}(r) + 2\ q_{24}(r)]$$

$$Q_{6,4} = \frac{n_{rx}^2}{n_{rz}^2}[g'_{42}(r) + 4\ g_{42}(r)] + \frac{n_{ry}^2}{n_{rz}^2}[p'_{42}(r) - p'_{22}(r) + 2\ p_{22}(r) + 4\ p_{42}(r)]$$

$$Q_{6,5} = \frac{n_{rx}^2}{n_{rz}^2}[g'_{24}(r) - 2\ g_{24}(r)] + \frac{n_{ry}^2}{n_{rz}^2}[p'_{44}(r) - p'_{24}(r) + 2\ p_{24}(r) + 4\ p_{44}(r)]$$

$$Q_{6,6} = \frac{n_{rx}^2}{n_{rz}^2}[g'_{46}(r) + 4\ g_{46}(r)] + \frac{n_{ry}^2}{n_{rz}^2}[p'_{46}(r) + 4\ p_{46}(r)]$$

$$Q_{6,7} = -\frac{n_{dx}^2}{n_{dz}^2}[h'_{20}(r) - 2\ h_{20}(r)] + \frac{n_{dy}^2}{n_{dz}^2}[s'_{20}(r) - 2\ s_{20}(r)]$$

$$Q_{6,8} = -\frac{n_{dx}^2}{n_{dz}^2}[h'_{22}(r) + h'_{42}(r) + 4\ h_{42}(r) - 2\ h_{22}(r)] - \frac{n_{dy}^2}{n_{dz}^2}[s'_{42}(r) + 4\ s_{42}(r)]$$

$$Q_{6,9} = -\frac{n_{dx}^2}{n_{dz}^2}[h'_{24}(r) + h'_{44}(r) + 4\ h_{44}(r) - 2\ h_{24}(r)] + \frac{n_{dy}^2}{n_{dz}^2}[s'_{24}(r) - 2\ s_{24}(r)]$$

$$Q_{6,a} = -\frac{n_{dx}^2}{n_{dz}^2}[k'_{42}(r) + 4\ g_{42}(r)] - \frac{n_{dy}^2}{n_{dz}^2}[r'_{42}(r) - r'_{22}(r) + 2\ r_{22}(r) + 4\ r_{42}(r)]$$

$$Q_{6,b} = -\frac{n_{dx}^2}{n_{dz}^2}\left[k'_{24}(r) - 2\ g_{24}(r)\right] - \frac{n_{dy}^2}{n_{dz}^2}\left[r'_{44}(r) - r'_{24}(r) + 2\ r_{24}(r) + 4\ r_{44}(r)\right]$$

$$Q_{6,c} = -\frac{n_{dx}^2}{n_{dz}^2}\left[k'_{46}(r) + 4\ g_{46}(r)\right] - \frac{n_{dy}^2}{n_{dz}^2}\left[r'_{46}(r) + 4\ r_{46}(r)\right]$$

$$Q_{7,1} = n_{rx}^2\ f_{20}(r) - n_{ry}^2\ q_{20}(r)$$

$$Q_{7,2} = n_{rx}^2\ [f_{22}(r) + f_{42}(r)] + n_{ry}^2\ q_{42}(r)$$

$$Q_{7,3} = n_{rx}^2\ [f_{24}(r) + f_{44}(r)] - n_{ry}^2\ q_{24}(r)$$

$$Q_{7,4} = n_{rx}^2\ g_{42}(r) + n_{ry}^2\ [p_{42}(r) - p_{22}(r)]$$

$$Q_{7,5} = n_{rx}^2\ g_{24}(r) + n_{ry}^2\ [p_{44}(r) - p_{24}(r)]$$

$$Q_{7,6} = n_{rx}^2\ g_{46}(r) + n_{ry}^2\ p_{46}(r)$$

$$Q_{7,7} = -\left[n_{dx}^2\ h_{20}(r) - n_{dy}^2\ s_{20}(r)\right]$$

$$Q_{7,8} = -\ n_{dx}^2\ [h_{22}(r) + h_{42}(r)] - n_{dy}^2\ s_{42}(r)$$

$$Q_{7,9} = -\ n_{dx}^2\ [h_{24}(r) + h_{44}(r)] + n_{dy}^2\ s_{24}(r)$$

$$Q_{7,a} = -\ n_{dx}^2\ g_{42}(r) - n_{dy}^2\ [r_{42}(r) - r_{22}(r)]$$

$$Q_{7,b} = -\ n_{dx}^2\ g_{24}(r) - n_{dy}^2\ [r_{44}(r) - r_{24}(r)]$$

$$Q_{7,c} = -\left[n_{dx}^2\ g_{46}(r) + n_{dy}^2\ r_{46}(r)\right]$$

$$Q_{8,1} = -\ f'_{20}(r) + q'_{20}(r) + 2\ f_{20}(r) - 2\ q_{20}(r)$$

$$Q_{8,2} = -\ f'_{22}(r) + f'_{42}(r) + q'_{42}(r) + 2\ f_{22}(r) + 4\ f_{42}(r) + 4\ q_{42}(r)$$

$$Q_{8,3} = -f'_{24}(r) + f'_{44}(r) + q'_{24}(r) + 2\ f_{24}(r) + 4\ f_{44}(r) - 2\ q_{24}(r)$$

$$Q_{8,4} = g'_{42}(r) + p'_{22}(r) + p'_{42}(r) + 4\ g_{42}(r) - 2\ p_{22}(r) + 4\ p_{42}(r)$$

$$Q_{8,5} = -g'_{24}(r) + p'_{24}(r) + p'_{44}(r) + 2\ g_{24}(r) - 2\ p_{24}(r) + 4\ p_{44}(r)$$

$$Q_{8,6} = g'_{46}(r) + p'_{46}(r) + 4\ g_{46}(r) + 4\ p_{46}(r)$$

$$Q_{8,7} = -[-h'_{20}(r) + s'_{20}(r) + 2\ h_{20}(r) - 2\ s_{20}(r)]$$

$$Q_{8,8} = -[-h'_{22}(r) + h'_{42}(r) + s'_{42}(r) + 2\ h_{22}(r) + 4\ h_{42}(r) + 4\ s_{42}(r)]$$

$$Q_{8,9} = -[-h'_{24}(r) + h'_{44}(r) + s'_{24}(r) + 2\ h_{24}(r) + 4\ h_{44}(r) - 2\ s_{24}(r)]$$

$$Q_{8,a} = -[k'_{42}(r) + r'_{22}(r) + r'_{42}(r) + 4\ g_{42}(r) - 2\ r_{22}(r) + 4\ r_{42}(r)]$$

$$Q_{8,b} = -[-k'_{24}(r) + r'_{24}(r) + r'_{44}(r) + 2\ g_{24}(r) - 2\ r_{24}(r) + 4\ r_{44}(r)]$$

$$Q_{8,c} = -[k'_{46}(r) + r'_{46}(r) + 4\ g_{46}(r) + 4\ r_{46}(r)]$$

$$Q_{9,1} = 0$$

$$Q_{9,2} = -f_{42}(r) + q_{42}(r)$$

$$Q_{9,3} = -f_{44}(r) + f_{64}(r) + q_{64}(r)$$

$$Q_{9,4} = -g_{42}(r) + p_{42}(r)$$

$$Q_{9,5} = g_{64}(r) + p_{44}(r) + p_{64}(r)$$

$$Q_{9,6} = -g_{46}(r) + p_{46}(r) + p_{66}(r)$$

$$Q_{9,7} = 0$$

$$Q_{9,8} = h_{42}(r) - s_{42}(r)$$

$$Q_{9,9} = h_{44}(r) - h_{64}(r) - s_{64}(r)$$

$$Q_{9,a} = g_{42}(r) - r_{42}(r)$$

$$Q_{9,b} = -[g_{64}(r) + r_{44}(r) + r_{64}(r)]$$

$$Q_{9,c} = g_{46}(r) - r_{46}(r) - r_{66}(r)$$

$$Q_{a,1} = 0$$

$$Q_{a,2} = \frac{n_{rx}^2}{n_{rz}^2}[f'_{42}(r) - 4 \; f_{42}(r)] + \frac{n_{ry}^2}{n_{rz}^2}[-q'_{42}(r) + 4 \; q_{42}(r)]$$

$$Q_{a,3} = \frac{n_{rx}^2}{n_{rz}^2}[f'_{44}(r) + f'_{64}(r) + 6 \; f_{64}(r) - 4 \; f_{44}(r)] + \frac{n_{ry}^2}{n_{rz}^2}[q'_{64}(r) + 6 \; q_{64}(r)]$$

$$Q_{a,4} = \frac{n_{rx}^2}{n_{rz}^2}[g'_{42}(r) - 4 \; g_{42}(r)] + \frac{n_{ry}^2}{n_{rz}^2}[-p'_{42}(r) + 4 \; p_{42}(r)]$$

$$Q_{a,5} = \frac{n_{rx}^2}{n_{rz}^2}[g'_{64}(r) + 6 \; g_{64}(r)] + \frac{n_{ry}^2}{n_{rz}^2}[p'_{64}(r) - p'_{44}(r) + 4 \; p_{44}(r) + 6 \; p_{64}(r)]$$

$$Q_{a,6} = \frac{n_{rx}^2}{n_{rz}^2}[g'_{46}(r) - 4 \; g_{46}(r)] + \frac{n_{ry}^2}{n_{rz}^2}[p'_{66}(r) - p'_{46}(r) + 4 \; p_{46}(r) + 6 \; p_{66}(r)]$$

$$Q_{a,7} = 0$$

$$Q_{a,8} = -\frac{n_{dx}^2}{n_{dz}^2}[h'_{42}(r) - 4 \; h_{42}(r)] - \frac{n_{dy}^2}{n_{dz}^2}[-s'_{42}(r) + 4 \; s_{42}(r)]$$

$$Q_{a,9} = -\frac{n_{dx}^2}{n_{dz}^2}\,[h'_{44}(r) + h'_{64}(r) + 6\ h_{64}(r) - 4\ h_{44}(r)] - \frac{n_{dy}^2}{n_{dz}^2}\,[s'_{64}(r) + 6\ s_{64}(r)]$$

$$Q_{a,a} = -\frac{n_{dx}^2}{n_{dz}^2}\,[k'_{42}(r) - 4\ g_{42}(r)] - \frac{n_{dy}^2}{n_{dz}^2}\,[-r'_{42}(r) + 4\ r_{42}(r)]$$

$$Q_{a,b} = -\frac{n_{dx}^2}{n_{dz}^2}\,[k'_{64}(r) + 6\ g_{64}(r)] - \frac{n_{dy}^2}{n_{dz}^2}\,[r'_{64}(r) - r'_{44}(r) + 4\ r_{44}(r) + 6\ r_{64}(r)]$$

$$Q_{a,c} = -\frac{n_{dx}^2}{n_{dz}^2}\,[k'_{46}(r) - 4\ g_{46}(r)] - \frac{n_{dy}^2}{n_{dz}^2}\,[r'_{66}(r) - r'_{46}(r) + 4\ r_{46}(r) + 6\ r_{66}(r)]$$

$$Q_{b,1} = 0$$

$$Q_{b,2} = n_{rx}^2\ f_{42}(r) - n_{ry}^2\ q_{42}(r)$$

$$Q_{b,3} = n_{rx}^2\ [f_{44}(r) + f_{64}(r)] + n_{ry}^2\ q_{64}(r)$$

$$Q_{b,4} = n_{rx}^2\ g_{42}(r) - n_{ry}^2\ p_{42}(r)$$

$$Q_{b,5} = n_{rx}^2\ g_{64}(r) + n_{ry}^2\ [-p_{44}(r) + p_{64}(r)]$$

$$Q_{b,6} = n_{rx}^2\ g_{46}(r) + n_{ry}^2\ [-p_{46}(r) + p_{66}(r)]$$

$$Q_{b,7} = 0$$

$$Q_{b,8} = -n_{dx}^2\ h_{42}(r) + n_{dy}^2\ s_{42}(r)$$

$$Q_{b,9} = -n_{dx}^2\ [h_{44}(r) + h_{64}(r)] - n_{dy}^2\ s_{64}(r)$$

$$Q_{b,a} = -n_{dx}^2\ g_{42}(r) + n_{dy}^2\ r_{42}(r)$$

$$Q_{b,b} = -n_{dx}^2\ g_{64}(r) - n_{dy}^2\ [-r_{44}(r) + r_{64}(r)]$$

$$Q_{b,c} = -n_{dx}^2\, g_{46}(r) - n_{dy}^2\, [-r_{46}(r) + r_{66}(r)]$$

$$Q_{c,1} = 0$$

$$Q_{c,2} = -f'_{42}(r) + q'_{42}(r) + 4\ f_{42}(r) - 4\ q_{42}(r)$$

$$Q_{c,3} = -f'_{44}(r) + f'_{64}(r) + q'_{64}(r) + 4\ f_{44}(r) + 6\ f_{64}(r) + 6\ q_{64}(r)$$

$$Q_{c,4} = -g'_{42}(r) + p'_{42}(r) + 4\ g_{42}(r) - 4\ p_{42}(r)$$

$$Q_{c,5} = g'_{64}(r) + p'_{44}(r) + p'_{64}(r) + 6\ g_{64}(r) - 4\ p_{44}(r) + 6\ p_{64}(r)$$

$$Q_{c,6} = -g'_{46}(r) + p'_{46}(r) + p'_{66}(r) + 4\ g_{46}(r) - 4\ p_{46}(r) + 6\ p_{66}(r)$$

$$Q_{c,7} = 0$$

$$Q_{c,8} = -[-h'_{42}(r) + s'_{42}(r) + 4\ h_{42}(r) - 4\ s_{42}(r)]$$

$$Q_{c,9} = -[-h'_{44}(r) + h'_{64}(r) + s'_{64}(r) + 4\ h_{44}(r) + 6\ h_{64}(r) + 6\ s_{64}(r)]$$

$$Q_{c,a} = -[-k'_{42}(r) + r'_{42}(r) + 4\ g_{42}(r) - 4\ r_{42}(r)]$$

$$Q_{c,b} = -[k'_{64}(r) + r'_{44}(r) + r'_{64}(r) + 6\ g_{64}(r) - 4\ r_{44}(r) + 6\ r_{64}(r)]$$

$$Q_{c,c} = -[-k'_{46}(r) + r'_{46}(r) + r'_{66}(r) + 4\ g_{46}(r) - 4\ r_{46}(r) + 6\ r_{66}(r)]$$

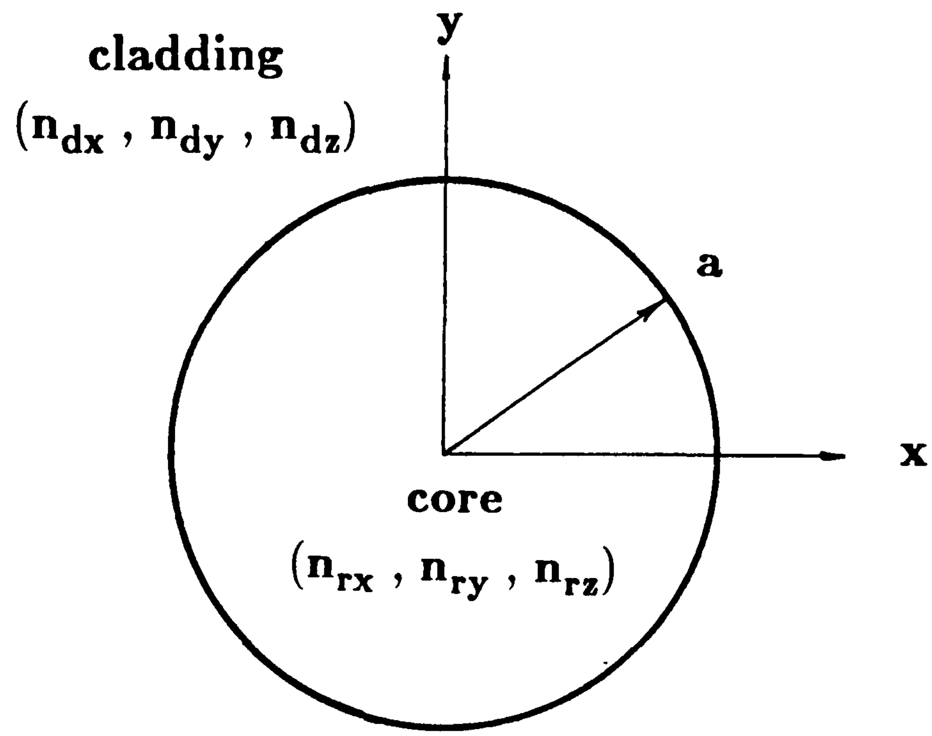
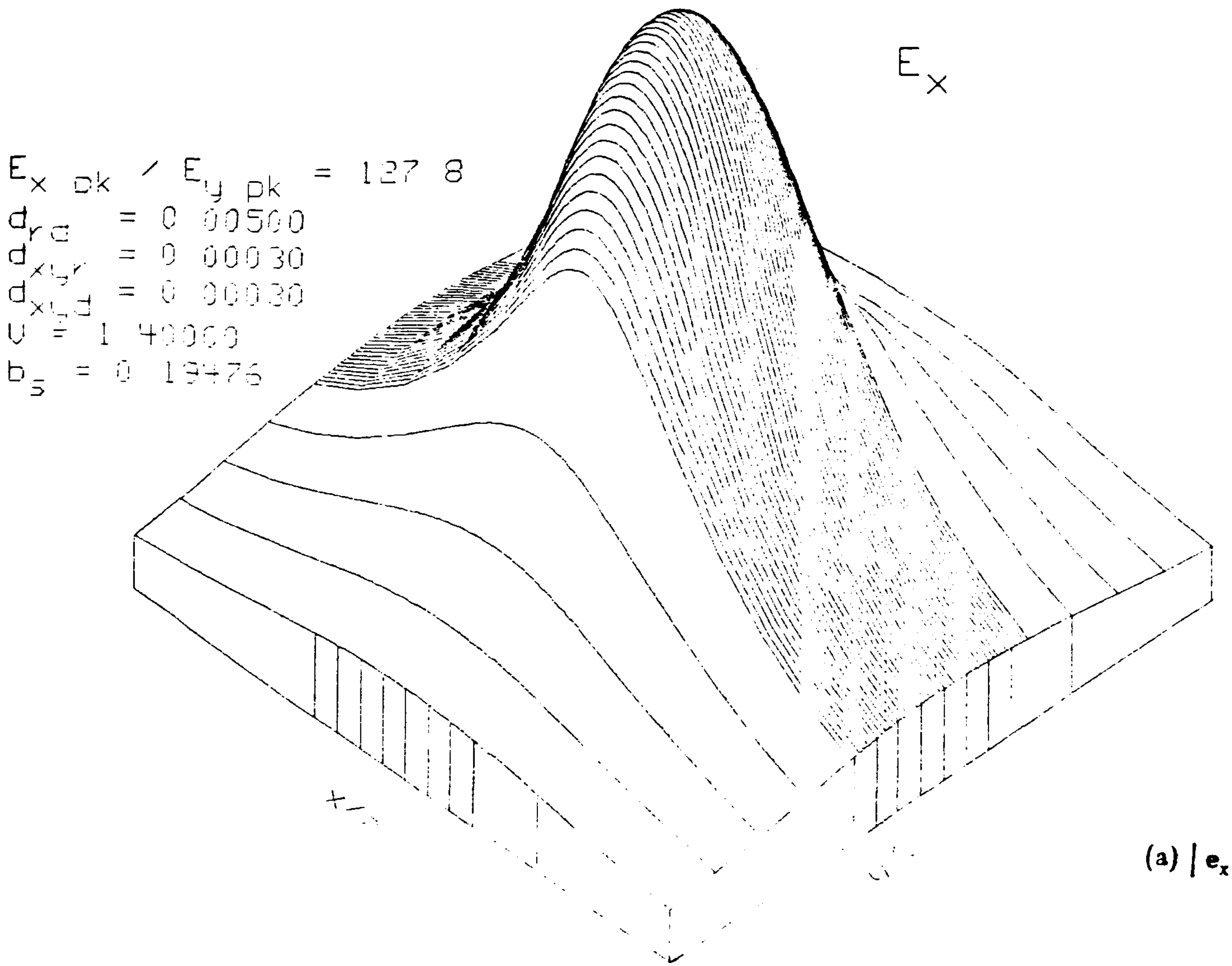
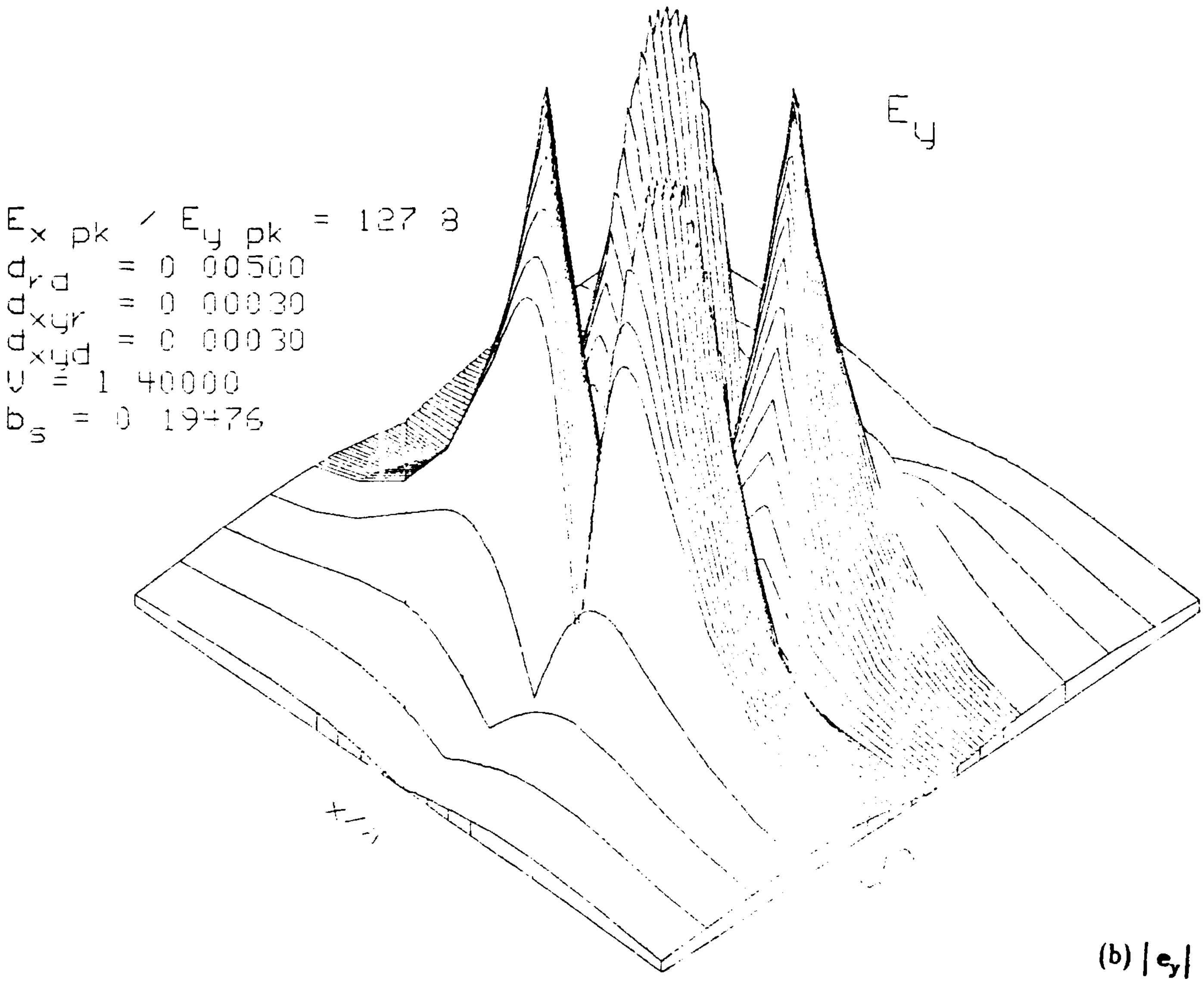


Figure 1. Biaxial fiber

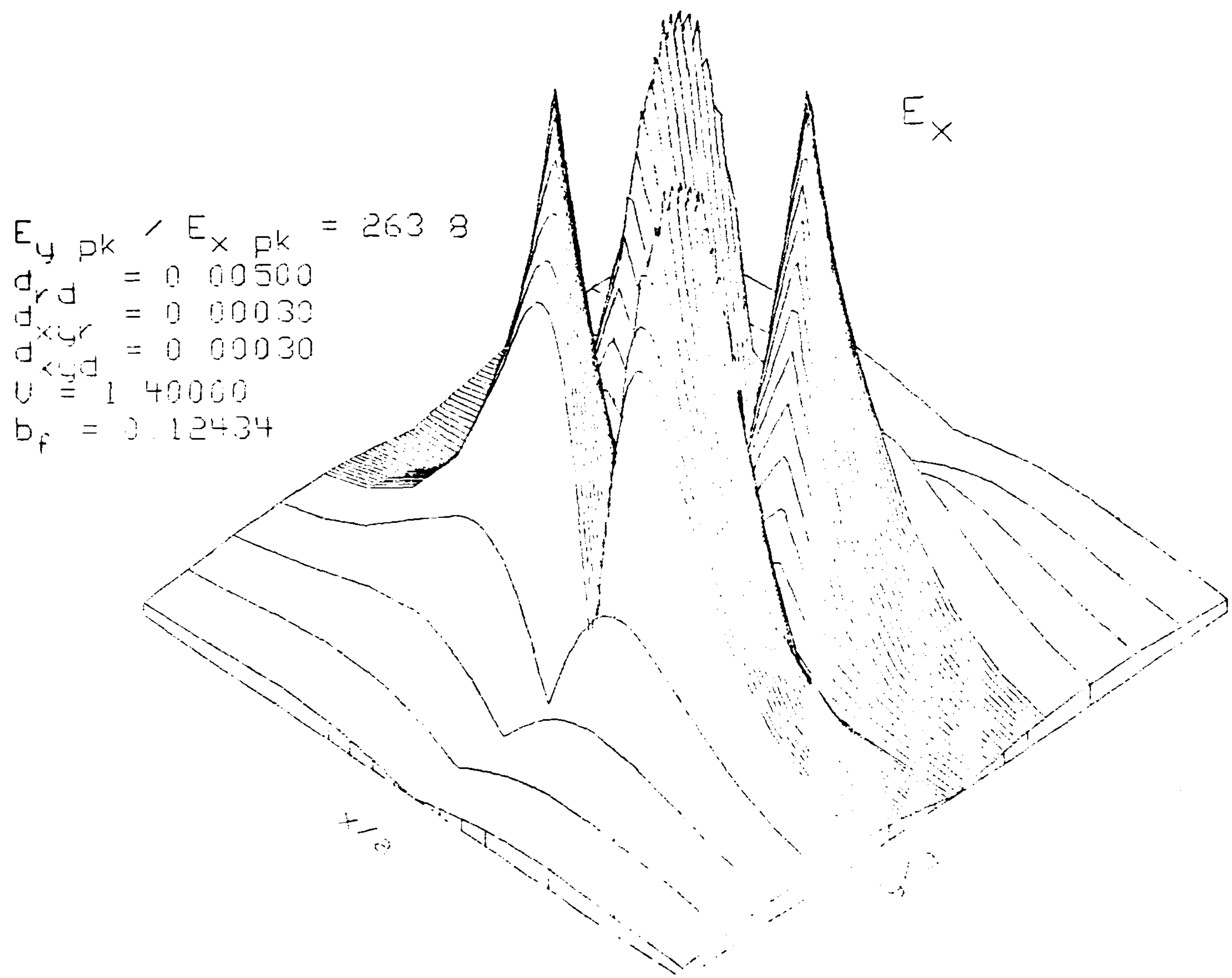


(a) $|\mathbf{e}_x|$

Figure 2. Electric fields of the lowest order polarization mode in a biaxial fiber.

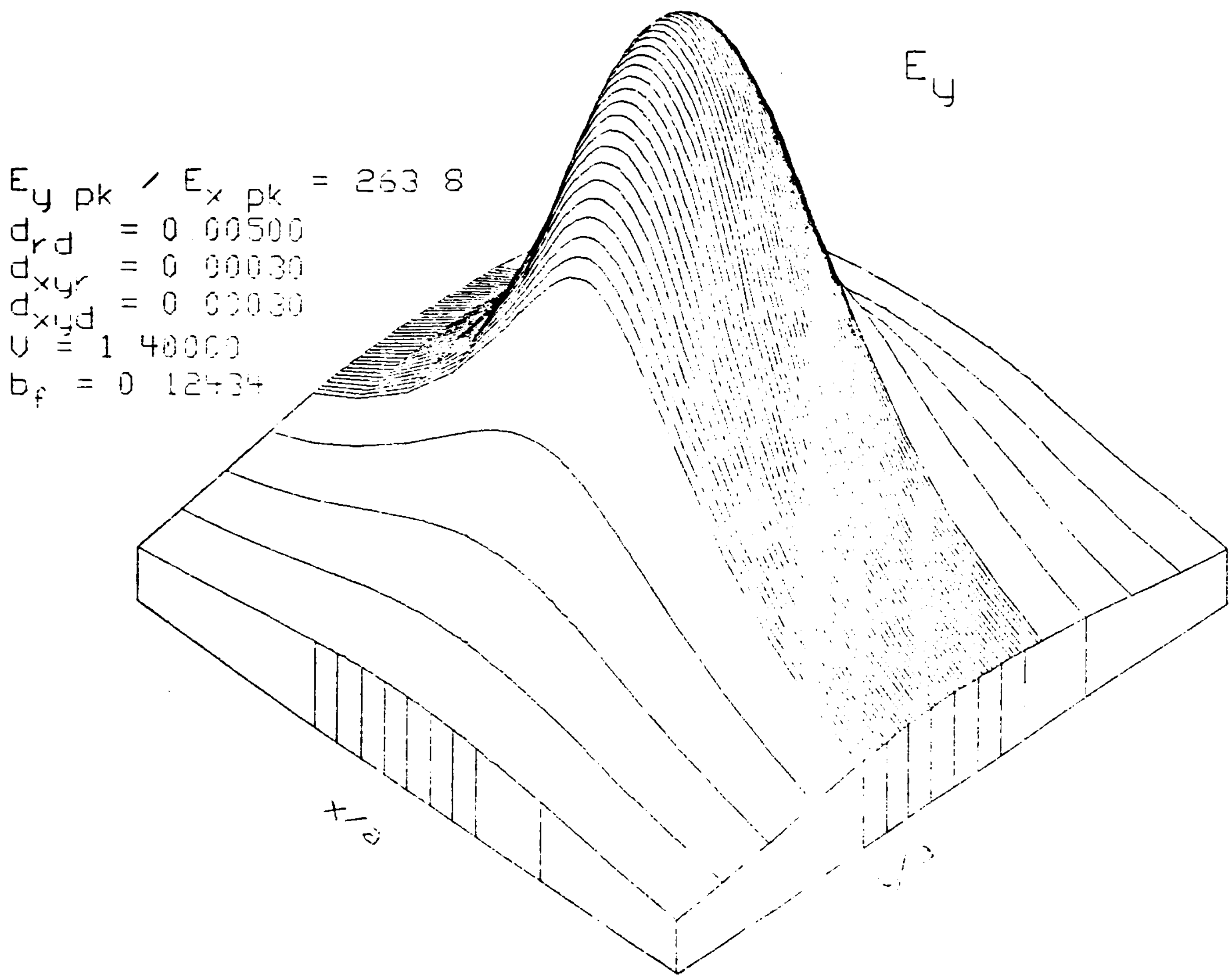

$E_{x\ pk}$ / $E_{y\ pk}$ = 127 8
d_{rd} = 0 00500
d_{xyr} = 0 00030
d_{xyd} = 0 00030
V = 1 40000
b_s = 0 19476
E_y

(b) $|e_y|$



(a) $|e_x|$

Figure 3. Electric fields of the second lowest order polarization mode in a biaxial fiber.



E_y
$E_{y\ pk} / E_{x\ pk}$ = 263 8
d_{rd} = 0 00500
d_{xyr} = 0 00030
d_{xyd} = 0 00030
V = 1 40000
b_f = 0 12434
x/a

(b) $|e_y|$

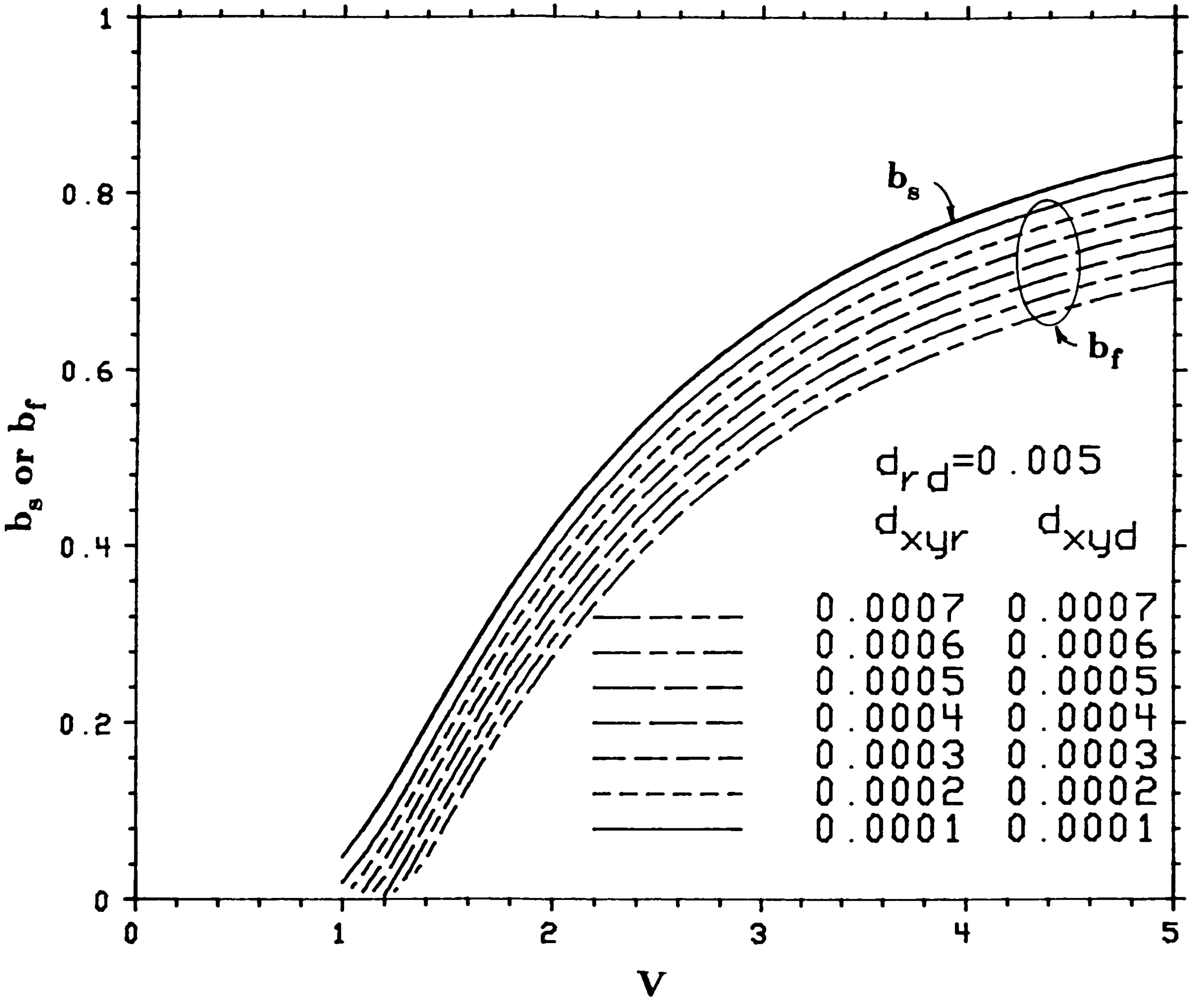


Figure 4. Dispersion of biaxial fibers

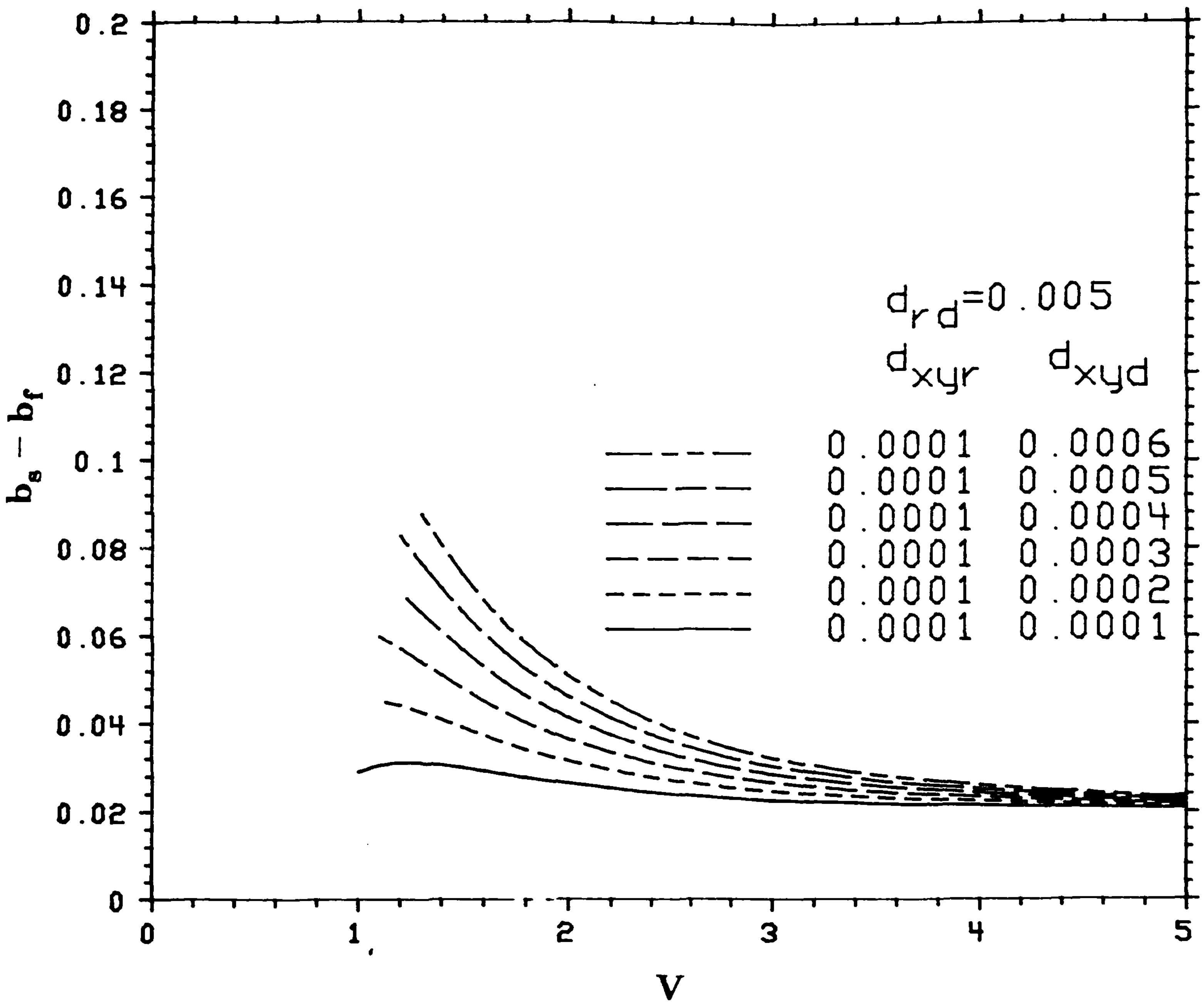


Figure 5. Birefringence of biaxial fibers with $d_{rd} = 0.005$ and $d_{xyr} = 0.0001$.

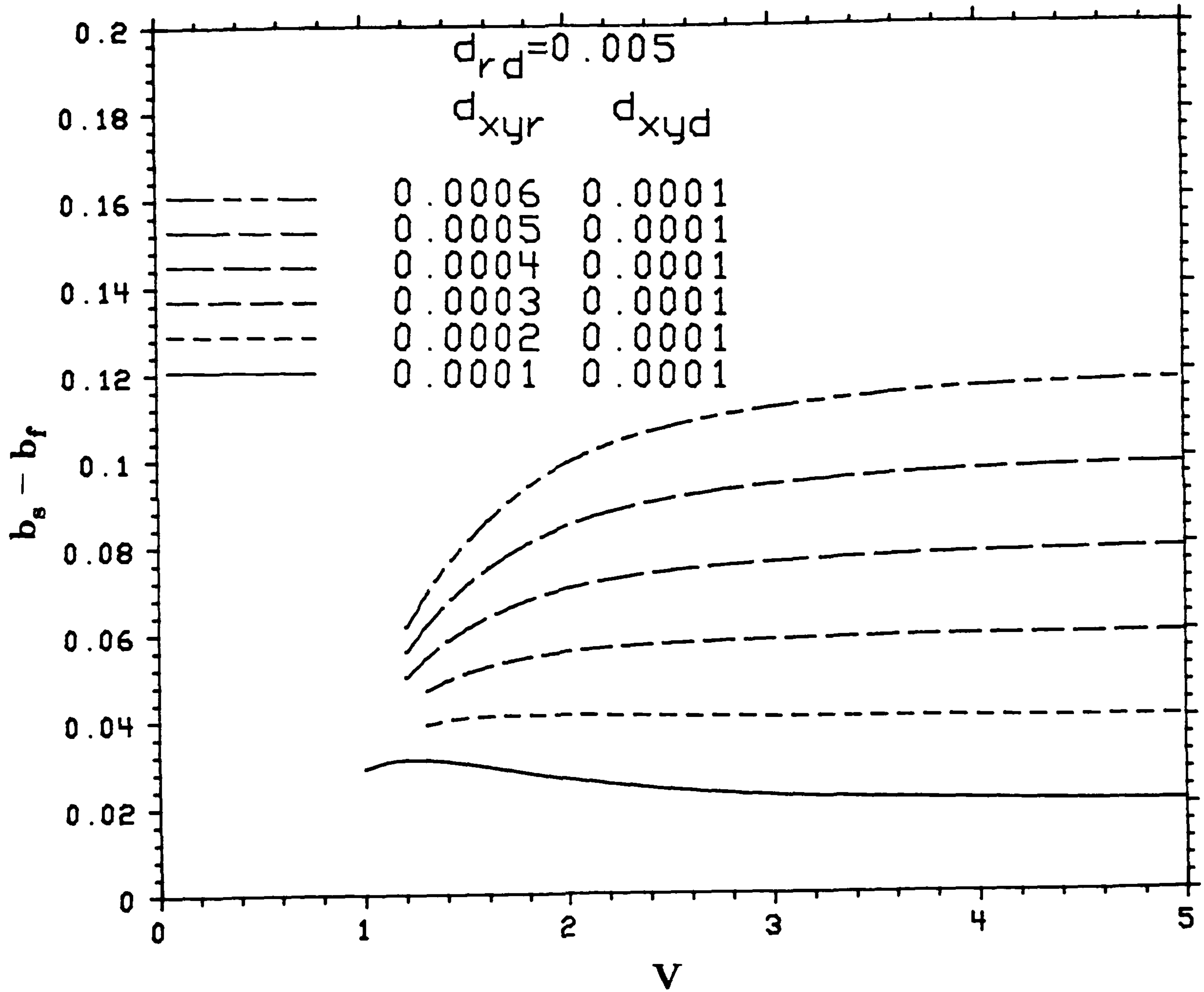


Figure 6. Birefringence of biaxial fibers with $d_{rd} = 0.005$ and $d_{xyd} = 0.0001$.

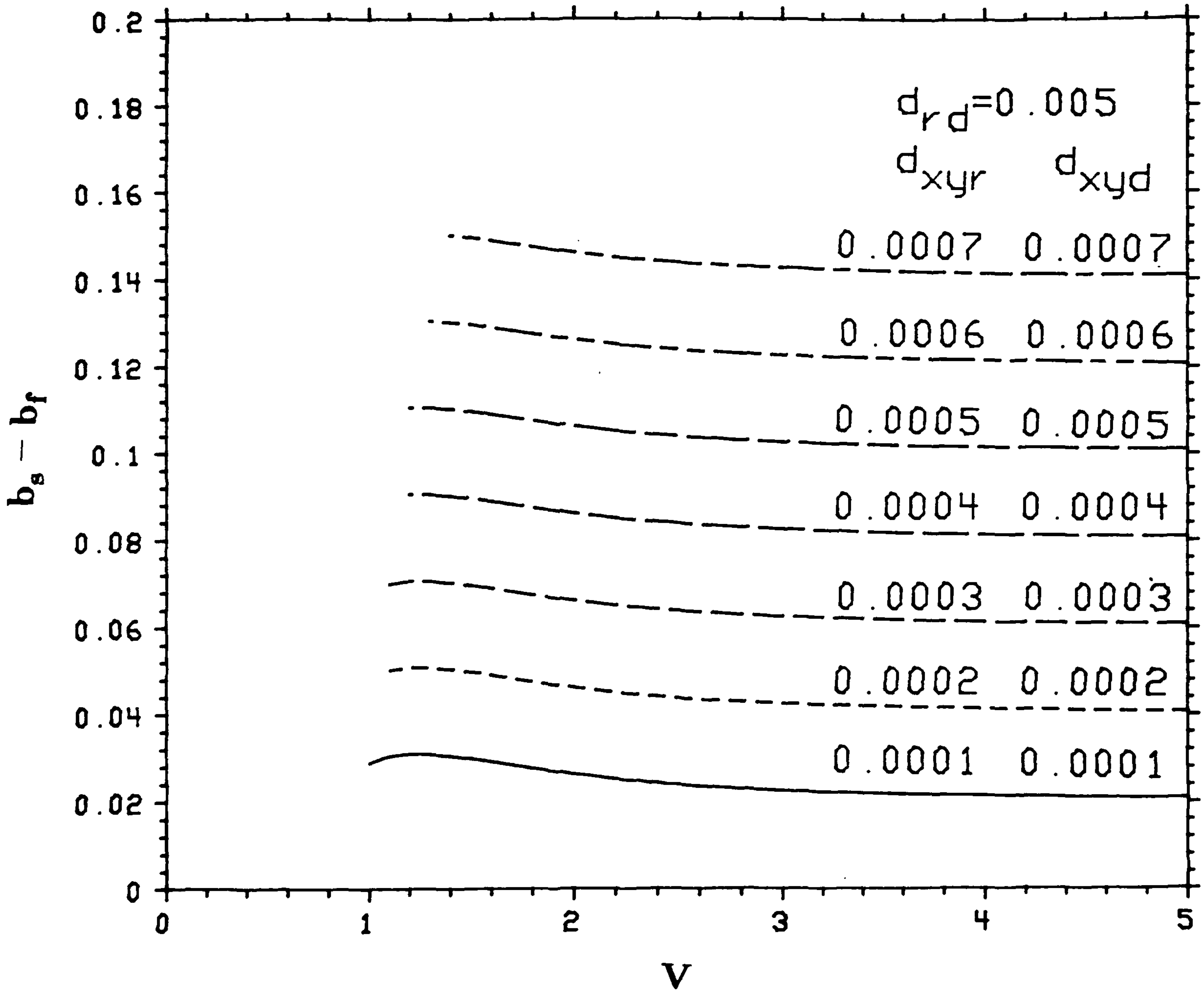


Figure 7. Birefringence of biaxial fibers with $d_{rd} = 0.005$ and $d_{xyr} = d_{xyd}$.

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